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Logarithm and its Applications

An approach to learn logarithm and its implementation in Mathematics

G. Tewani

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CHAPTER 1

Introduction to Logarithm

Before we can deal with logarithms, we need to revise indices. This is because logarithms and indices are closely related, and in order to understand logarithms, a good knowledge of indices is required.

We know that $16 = 2^4$

Here, the number 4 is the power. Sometimes we call it an exponent. Sometimes we call it an index. In the expression 2^4, the number 2 is called the base.

Why Do We Study Logarithms?

In order to motivate our study of logarithms, consider the following:

We know that $16 = 2^4$. We also know that $8 = 2^3$

Suppose that we wanted to multiply 16 by 8. One way is to carry out the multiplication directly using long-multiplication and obtain 128. But this could be long and tedious if the numbers were larger than 8 and 16. Can we do this calculation another way using the powers? Note that 16×8 can be written as $2^4 \times 2^3$. This equals 2^7. Using the rules of indices which tell us to add the powers 4 and 3 to give the new power 7, a multiplication sum has been reduced to an addition sum. Similarly, we can divide 16 by 8. $16 \div 8$ can be written $2^4 \div 2^3$. This equals 2^1 or simply 2. Using the rules of indices which tell us to subtract the powers 4 and 3 to give the new power, 1. If we had a look-up table containing powers of 2, it would be straightforward to look up 2^7 and obtain $2^7 = 128$ as the result of finding 16×8. Note that by using the powers, we have changed a multiplication problem into one involving addition (the addition of the powers, 4 and 3). Historically, this observation led *John Napier (1550-1617) and Henry Briggs (1561-1630)* to develop logarithms as a way of replacing multiplication with addition, and also division with subtraction.

What is a Logarithm?

Consider the expression $16 = 2^4$. Remember that 2 is the base, and 4 is the power. An alternative, yet equivalent, way of writing this expression is $\log_2 16 = 4$. This is stated as 'log to base 2 of 16 equals 4'. We see that the logarithm is the same as the

power or index in the original expression. It is the base in the original expression which becomes the base of the logarithm. The two statements $16 = 2^4$ and $\log_2 16 = 4$ are equivalent statements. If we write either of them, we are automatically implying the other.

Thus logarithm is the inverse operation to exponentiation. The logarithm of a number to a given base is the exponent to which the base must be raised in order to produce that number.

For example, the logarithm of 1000 to base 10 is 3, because 10 to the power of 3 is 1000, i.e., $10^3 = 1000$. We write $\log_{10}1000 = 3$. Here '10' is called the base of the logarithm.

In general, if $x = a^n$ then equivalently $n = \log_a x$. Note in $x = a^n$ the restriction on the base is that it is not '0' or '1' as $0^n = 0$ and $1^n = 1$ and hence are constant and won't have many of the same properties that general exponential relations have. Also, we avoid negative numbers as base. For instance, if we allow value of base '–4' then we have $(-4)^n$ for which $(-4)^{0.5}$ is a complex number. We only want real numbers to arise from such power calculations, so we require that base is not a negative number. Also, in $x = a^n$, n can take any real value, so $\log_a x$ can also take any real value. Also, $x = a^n > 0$, so $\log_a x$ is defined only if $x > 0$.

Note: • Since $a^0 = 1$ and $a^1 = a$, where a is any positive real number, we have $\log_a 1 = 0$ and $\log_a a = 1$.

• $a^n = (a^n)$, so we have $\log_a(a^n) = n$

Example 1 Find the value of each of the following:

(i) $\log_9 81$ (ii) $\log_{\sqrt{2}} 4$ (iii) $\log_{2\sqrt{3}} 1728$

(iv) $\log_{(\tan 40^\circ)}(\cot 50^\circ)$ (v) $\log_{2.25} 0.\overline{4}$ (vi) $\log_{(\sqrt{2}+\sqrt{3})}(5+2\sqrt{6})$

Sol. (i) $\log_9 81 = x$

$\therefore \quad 81 = 9^x$

$\therefore \quad 9^2 = 9^x$

$\therefore \quad x = 2$

(ii) $\log_{\sqrt{2}} 4 = x$

$\therefore \quad 4 = (\sqrt{2})^x$

$\therefore \quad 2^2 = 2^{x/2}$

$\therefore \quad x/2 = 2$

$\therefore \quad x = 4$

(iii) $\log_{2\sqrt{3}} 1728 = x$

$\therefore \quad 1728 = \left(2\sqrt{3}\right)^x$

$\therefore \quad (2^6 3^3) = \left(2\sqrt{3}\right)^x$

$\therefore \quad \left(2\sqrt{3}\right)^6 = \left(2\sqrt{3}\right)^x$

$\therefore \quad x = 6$

(iv) $\log_{(\tan 40°)}(\cot 50°) = x$

$\therefore \quad (\tan 40°) = (\cot 50°)^x$

$\therefore \quad (\cot 50°) = (\cot 50°)^x$

$\therefore \quad x = 1$

(v) $x = 0.\overline{4} = 0.4444444\ldots$

$\therefore \quad 10x = 4.4444444\ldots$

Subtracting, we get $9x = 4$

$\therefore \quad x = 4/9$

Now, $\log_{2.25} 0.\overline{4} = \log_{2.25} \dfrac{4}{9} = x$

$\therefore \quad \dfrac{4}{9} = (2.25)^x$

$\therefore \quad (2.25)^{-1} = (2.25)^x$

$\therefore \quad x = -1$

(vi) $\log_{(\sqrt{2}+\sqrt{3})}(5+2\sqrt{6}) = x$

$\therefore \quad (5+2\sqrt{6}) = (\sqrt{2}+\sqrt{3})^x$

$\therefore \quad (\sqrt{2}+\sqrt{3})^2 = (\sqrt{2}+\sqrt{3})^x$

$\therefore \quad x = 2$

Example 2 Find the value of x in each of the following cases:

(i) $\log_2 x = 3$ (ii) $\log_9 x = 2.5$ (iii) $\log_x 81 = 4$

(iv) $2^x = 7$ (v) $10^{2x-1} = 17$

Sol. (i) $\log_2 x = 3$

$\therefore \quad x = 2^3 = 8$

(ii) $\log_9 x = 2.5$

$\therefore \quad x = 9^{2.5} = (3^2)^{2.5} = 3^5 = 243$

(iii) $\log_x 81 = 4$

$\therefore \quad 81 = x^4$

$\therefore \quad x = 3$

(iv) $2^x = 7$

$\therefore \quad x = \log_2 7$

(v) $10^{2x-1} = 17$

$\therefore \quad 2x - 1 = \log_{10} 17$

$\therefore \quad 2x = \log_{10} 17 + 1$

$\therefore \quad x = \dfrac{1+\log_{10} 17}{2}$

Example 3 Find the value of $3^{2\log_9 3}$.

Sol. $3^{2\log_9 3}$

$= 3^{2\log_9 9^{1/2}}$

$= 3^{2\left(\frac{1}{2}\right)}$

$= 3$

Example 4 Find the value of $\log_5 \log_2 \log_3 \log_2 512$.

Sol. $\log_5 \log_2 \log_3 \log_2(2^9)$

$= \log_5 \log_2 \log_3 9$

$= \log_5 \log_2 \log_3 3^2$

$= \log_5 \log_2 2$

$= \log_5 1$

$= 0$

Example 5 Find the value of $\log_{1/3} \sqrt[4]{729.\sqrt[3]{9^{-1}.27^{-4/3}}}$.

Sol. $\log_{1/3} \sqrt[4]{729.\sqrt[3]{9^{-1}.27^{-4/3}}}$

$= \log_{1/3} \sqrt[4]{729.\sqrt[3]{3^{-2}.3^{-4}}}$

$= \log_{1/3} \sqrt[4]{3^6 \cdot 3^{-2}}$

$= \log_{1/3} 3$

$= -1$

Example 6 Prove that $\frac{2}{5} < \log_{10} 3 < \frac{1}{2}$.

Sol. Let $\log_{10} 3 > \frac{2}{5}$

$\Rightarrow \quad 3 > 10^{\frac{2}{5}}$

$\Rightarrow \quad 3^5 > 10^2$, which is true

Now $\quad \log_{10} 3 < \frac{1}{2}$

$\Rightarrow \quad 3 < 10^{\frac{1}{2}}$

$\Rightarrow \quad 3^2 < 10$, which is true

Hence $\frac{2}{5} < \log_{10} 3 < \frac{1}{2}$

Example 7. Arrange $\log_2 5$, $\log_{0.5} 5$, $\log_7 5$, $\log_3 5$ in increasing order.

Sol. $\log_2 5$ = exponent of 2 for which we get 5

$\log_7 5$ = exponent of 7 for which we get 5

Clearly $\log_2 5 > \log_7 5$

With similar reasons we have

$\log_7 5 < \log_3 5 < \log_2 5 < \log_{0.5} 5$

Example 8. Prove that number $\log_2 7$ is an irrational number.

Sol. Let $\log_2 7$ is a rational number.

$$\Rightarrow \quad \log_2 7 = \frac{p}{q} \Rightarrow 7 = 2^{\frac{p}{q}}$$

$\Rightarrow \quad 7^q = 2^p$ which is not possible for any integral values of p and q.

Hence, $\log_2 7$ is not rational.

Example 9. Which of the following numbers are positive/negative?

(i) $\log_2 7$ (ii) $\log_{0.2} 3$ (iii) $\log_{1/3}(1/5)$

(iv) $\log_4 3$ (v) $\log_2(\log_2 9)$

Sol. (i) Let $\log_2 7 = x \Rightarrow 7 = 2^x \Rightarrow x > 0$

(ii) Let $\log_{0.2} 3 = x \Rightarrow 3 = 0.2^x \Rightarrow x < 0$

(iii) Let $\log_{1/3}(1/5) = x \Rightarrow 1/5 = (1/3)^x \Rightarrow 5 = 3^x \Rightarrow x > 0$

(iv) Let $\log_4 3 = x \Rightarrow 3 = 4^x \Rightarrow x < 0$

(v) Let $\log_2(\log_2 9) = x \Rightarrow \log_2 9 = 2^x \Rightarrow 9 = 2^{2^x} \Rightarrow x > 0$

Example 10. If $\log_a 3 = 2$ and $\log_b 8 = 3$, then prove that $\log_a b = \log_3 4$.

Sol. If $\log_a 3 = 2$

$\Rightarrow \quad 3 = a^2$

Also, $\log_b 8 = 3$

$\Rightarrow \quad b = 2$

$\Rightarrow \quad \log_a b = \log_{\sqrt{3}} 2 = x$ (let)

$\Rightarrow \quad 2 = \left(\sqrt{3}\right)^x$

$\Rightarrow \quad 4 = 3^x$

$\Rightarrow \quad x = \log_3 4$

Example 11. If $\log_3 y = x$ and $\log_2 z = x$, find 72^x in terms of y and z.

Sol. $\log_3 y = x$

$\therefore \quad y = 3^x$

$\log_2 z = x$

$\therefore \quad z = 2^x$

Now $72^x = (2^3 3^2)^x = 2^{3x} 3^{2x} = (2^x)^3 (3^x)^2 = y^3 z^2$

Example 12. Solve for x : $\log_4 \log_3 \log_2 x = 0$

Sol. $\log_4 \log_3 \log_2 x = 0$

$\therefore \quad \log_3 \log_2 x = 1$

$\therefore \quad \log_2 x = 3$

$\therefore \quad x = 2^3 = 8$

Example 13. If $b > 1$, $\sin t > 0$, $\cos t > 0$ and $\log_b(\sin t) = x$ then prove that $\log_b(\cos t) = \frac{1}{2}\log_b(1 - b^{2x})$.

Sol. $\log_b \sin t = x \Rightarrow \sin t = b^x$

Let $\log_b(\cos t) = y$

$\Rightarrow \quad b^y = \cos t$

$\Rightarrow \quad b^{2y} = \cos^2 t = 1 - \sin^2 t = 1 - b^{2x}$

$\Rightarrow \quad 2y = \log_b(1 - b^{2x})$

$\Rightarrow \quad y = \frac{1}{2}\log_b(1 - b^{2x})$

Example 14 If $10^{\log_p(\log_q(\log_r x))} = 1$ and $\log_q(\log_r(\log_p x)) = 0$ then prove that $p = r^{q/r}$.

Sol. $10^{\log_p(\log_q(\log_r x))} = 1$

$\Rightarrow \quad \log_p(\log_q(\log_r x)) = 0$

$\Rightarrow \quad \log_q(\log_r x) = 1$

$\Rightarrow \quad \log_r x = q$

$\Rightarrow \quad x = r^q$ (1)

$\log_q(\log_r(\log_p x)) = 0$

$\Rightarrow \quad \log_r(\log_p x) = 1$

$\Rightarrow \quad \log_p x = r$

$\Rightarrow \quad x = p^r$ (2)

From (1) and (2), $r^q = p^r$

$\Rightarrow \quad p = r^{q/r}$

Example 15 If $\log_a x = b$ for permissible values of a and x then identify the statement(s) which can be correct?

(a) If a and b are two irrational numbers then x can be rational.

(b) If a is rational and b is irrational then x can be rational.

(c) If a is irrational and b is rational then x can be rational.

(d) If a is rational and b is rational then x can be rational.

Sol. **(a, b, c, d)**

$$\log_a x = b \Rightarrow x = a^b$$

(a) for $a = \sqrt{2}^{\sqrt{2}} \notin Q$ and $b = \sqrt{2} \notin Q$; $x = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ which is rational

(b) For $a = 2 \in Q$ and $b = \log_2 3 \notin Q$; $x = 3$ which is rational.

(c) For $a = \sqrt{2}$ and $b = 2$; $x = 2$

(d) It is obviously correct.

Exercise 1

1. Find the value of each of the following:
   (i) $\log_{10} 0.001$ (ii) $\log_2(1/32)$ (iii) $\log_{9\sqrt{3}} 0.\bar{1}$
   (iv) $\log_{(5+2\sqrt{6})}(5-2\sqrt{6})$
2. Find the value of x in each of the following cases:
   (i) $\log_{81} x = \dfrac{3}{2}$ (ii) $\log_{\sqrt{2}} x = 4$ (iii) $\log_2 4x = 5$
3. Find the value of the following:
   (i) $\log_{1/4}\left(\dfrac{1}{16}\right)^{-2}$ (ii) $\sqrt{(\log_{0.5} 4)^2}$ (iii) $\dfrac{\log_2 32}{\log_3 \sqrt{243}}$
4. Find the value of $\log_3\left(\tan\dfrac{7\pi}{6}\right) + \log_{\cot(4\pi/3)}(3)$.
5. Find the value of $\log_2 \log_2 \log_4 256 + 2\log_{\sqrt{2}} 2$.
6. If $\log_{\sqrt{8}} b = 3\dfrac{1}{3}$, then find the value of b.
7. Find the value of log tan 1° log tan 2° .... log tan 89°.
8. Prove that $\log_4 18$ is an irrational number.
9. Which one of the following is the smallest?
   (a) $\log_{10}\pi$ (b) $\sqrt{\log_{10}\pi^2}$
   (c) $\left(\dfrac{1}{\log_{10}\pi}\right)^3$ (d) $\dfrac{1}{\log_{10}\sqrt{\pi}}$
10. Which of the following numbers are positive/negative?
    (i) $\log_{\sqrt{3}}\sqrt{2}$ (ii) $\log_{1/7}(2)$ (iii) $\log_{1/3}(1/5)$
    (iv) $\log_3(4)$ (v) $\log_7(2.11)$ (vi) $\log_3(\sqrt{7}-2)$
    (vii) $\log_4\left(\dfrac{\sqrt{2}+1}{\sqrt{2}-1}\right)$ (viii) $\log_3\left(\dfrac{2\cdot\sqrt[3]{3}}{3}\right)$
11. If $\log_5 x = a$ and $\log_2 y = a$, find 100^{2a-1} in terms of x and y.
12. If $\log_x \log_{18}\left(\sqrt{2}+\sqrt{8}\right) = -\dfrac{1}{2}$ then the value of x is

Exercise 2

1. Find the value of x satisfying $\sqrt{3}^{-4+2\log_{\sqrt{5}} x} = \dfrac{1}{9}$.

2. Find the value of $\log_{\frac{9}{4}}\left(\frac{1}{2\sqrt{3}}\sqrt{6-\frac{1}{2\sqrt{3}}\sqrt{6-\frac{1}{2\sqrt{3}}\sqrt{6-\frac{1}{2\sqrt{3}}\ldots\infty}}}\right)$.

3. Find the value of $\log_2\left(2\sqrt[3]{9}-2\right)\left(12\sqrt[3]{3}+4+4\sqrt[3]{9}\right)$.

4. If $\log_4 5 = a$ and $\log_5 6 = b$ then prove that $\log_3 2 = \frac{1}{2ab-1}$.

5. If $\log_2\left(\log_2(\log_3 x)\right) = \log_2\left(\log_3(\log_2 y)\right) = 0$ then find the value of $(x-y)$.

6. If $\log_{175}5x = \log_{343}7x$, then find the value of $\log_{42}(x^4 - 2x^2 + 7)$.

7. If $\log_4 A = \log_6 B = \log_9(A+B)$ then find the value of $\frac{B}{A}$.

8. Find the value of x satisfying the equation $10^x + 10^{-x} = 4$.

9. If $\log_b n = 2$ and $\log_n 2b = 2$, then find the value of b.

10. If $\frac{\log_2 x}{4} = \frac{\log_2 y}{6} = \frac{\log_2 z}{3k}$ and $x^3y^2z = 1$, then find the value of k.

11. Let S be the set of ordered triples (x, y, z) of real numbers for which $\log_{10}(x+y) = z$ and $\log_{10}(x^2+y^2) = z+1$. Suppose there are real numbers a and b such that for all ordered triples (x, y, z) in S we have $x^3+y^3 = a\cdot 10^{3z} + b\cdot 10^{2z}$. Then find the value of $(a+b)$.

CHAPTER

2 Laws Related to Logarithm

Law 1: For $m, n > 0$; $\log_a (mn) = \log_a m + \log_a n$

Proof : Let $\log_a m = x$ and $\log_a n = y$.

Then $\log_a m = x \Rightarrow a^x = m$ and $\log_a n = y \Rightarrow a^y = n$

$\therefore \quad mn = a^x . a^y$

$\Rightarrow \quad mn = a^{x+y}$

$\Rightarrow \quad \log_a(mn) = x + y$

$\Rightarrow \quad \log_a(mn) = \log_a m + \log_a n$

In general for positive rational numbers, $x_1, x_2, \ldots x_n$:

$\log_a(x_1 x_2 \cdots x_n) = \log_a x_1 + \log_a x_2 + \ldots + \log_a x_n$

Law 2: For $m, n > 0$; $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

Proof: Let $\log_a m = x \Rightarrow a^x = m$ and $\log_a n = y \Rightarrow a^y = n$.

$$\therefore \quad \frac{m}{n} = \frac{a^x}{a^y} \Rightarrow \frac{m}{n} = a^{x-y} \Rightarrow \log_a\left(\frac{m}{n}\right) = x - y$$

$$\Rightarrow \quad \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Law 3: For $m, n > 0$ $\log_a(m^n) = n. \log_a m$

Proof: Let $\log_a m = x \Rightarrow a^x = m$.

$$\Rightarrow \quad (a^x)^n = m^n$$

$$\Rightarrow \quad a^{xn} = m^n$$

$$\Rightarrow \quad \log_a(m^n) = nx$$

$$\Rightarrow \quad \log_a(m^n) = n. \log_a m$$

Law 4: $\log_{(a^p)} n = \frac{1}{p} \log_a n$

Proof : Let $\log_{(a^p)} n = x$

Then, $(a^p)^x = n$

$\Rightarrow \quad a^{px} = n$

$\Rightarrow \quad px = \log_a n$

$\Rightarrow \quad x = \frac{1}{p}\log_a n$

$\Rightarrow \quad \log_{(a^p)} n = \frac{1}{p}\log_a n$

Law 5: $\log_a m = \frac{1}{\log_m a}$

Proof: Let $\log_a m = x$

$\Rightarrow \quad m = a^x$

$\Rightarrow \quad m^{\frac{1}{x}} = a$

$\Rightarrow \quad \frac{1}{x} = \log_m a$

$\Rightarrow \quad x = \frac{1}{\log_m a}$

$\Rightarrow \quad \log_a m = \frac{1}{\log_m a}$

Example 1 Evaluate each of the following:

(i) $\log_{10}500 - \log_{10}5$

(ii) $4\log_{10}5 + 2\log_{10}4$

(iii) $\log_{10}6 + 2\log_{10}5 + \log_{10}4 - \log_{10}3 - \log_{10}2$

Sol. (i) $\log_{10}500 - \log_{10}5$

$= \log_{10}(500 \div 5)$

$= \log_{10}100$

$= 2$

(ii) $4\log_{10}5 + 2\log_{10}4$

$= \log_{10}5^4 + \log_{10}4^2$

$= \log_{10}5^4 + \log_{10}2^4$

$= \log_{10}(5^4 \times 2^4)$

$= \log_{10}(10^4)$

$= 4$

(iii) $\log_{10} 6 + 2 \log_{10} 5 + \log_{10} 4 - \log_{10} 3 - \log_{10} 2$

$= \log_{10} 6 + \log_{10} 5^2 + \log_{10} 4 - \log_{10} 3 - \log_{10} 2$

$$= \log_{10} \frac{6 \times 5^2 \times 4}{3 \times 2}$$

$= \log_{10}(5^2 \times 4)$

$= \log_{10} 100$

$= 2$

Example 2 Find the value of the following.

(i) $\log\left(\frac{9}{14}\right) + \log\left(\frac{35}{24}\right) - \log\left(\frac{15}{16}\right)$

(ii) $7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80}$

(iii) $\log_{10}5 \,.\, \log_{10}20 + (\log_{10}2)^2$

Sol. (i) $\log\left(\frac{9}{14}\right) + \log\left(\frac{35}{24}\right) - \log\left(\frac{15}{16}\right)$

$$= \log\left(\frac{9}{14} \times \frac{35}{24} \times \frac{16}{15}\right)$$

$$= \log\left(\frac{3^2}{2 \times 7} \times \frac{5 \times 7}{2^3 \times 3} \times \frac{2^4}{3 \times 5}\right)$$

$= \log 1$

$= 0$

(ii) $7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80}$

$$= \log\left(\frac{16}{15}\right)^7 \left(\frac{25}{24}\right)^5 \left(\frac{81}{80}\right)^3$$

$$= \log\left(\frac{2^{28}}{3^7 5^7}\right)\left(\frac{5^{10}}{2^{15} 3^5}\right)\left(\frac{3^{12}}{2^{12} 5^3}\right)$$

$= \log 2$

(iii) $\log_{10}\left(\frac{10}{2}\right) . \log_{10}(10 \times 2) + (\log_{10}2)^2$
$= (1 - \log_{10}2)(1 + \log_{10}2) + (\log_{10}2)^2 = 1$

Example 3 Prove that $\log_7 \log_7 \sqrt{7\sqrt{(7\sqrt{7})}} = 1 - 3\log_7 2$.

Sol. $\log_7 \log_7 \sqrt{7\sqrt{(7\sqrt{7})}}$

$$= \log_7 \log_7 7^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}$$

$$= \log_7\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right) = \log_7\left(\frac{7}{8}\right) = 1 - \log_7 8 = 1 - 3\log_7 2$$

Example 4 Prove that $\log ab - \log|b| = \log|a|$

Sol. $\log ab$ is defined if $ab > 0$ or a and b have same sign.

Case (i) : $a, b > 0$

$\Rightarrow \quad \log ab - \log|b| = \log a + \log b - \log b = \log a$ (i)

Case (ii) : $a, b < 0$

$\Rightarrow \quad \log ab - \log|b| = \log(-a) + \log(-b) - \log(-b) = \log(-a)$ (ii)

From (i) and (ii), we have $\log ab - \log|b| = \log|a|$

Example 5 If $\log_7 2 = m$, then find $\log_{49} 28$ in terms of m.

Sol. $\log_{49} 28 = \log_{(7^2)}\left(2^2 \times 7\right)$

$$= \frac{1}{2}\log_7\left(2^2 \times 7\right)$$

$$= \frac{1}{2}\left[\log_7 2^2 + \log_7 7\right]$$

$$= \frac{1}{2}\left[2\log_7 2 + 1\right] = \frac{1+2m}{2}$$

Example 6 If $a^2 + b^2 = 7ab$, prove that $\log\left(\frac{a+b}{3}\right) = \frac{1}{2}(\log a + \log b)$.

Sol. $a^2 + b^2 = 7ab$

$\Rightarrow \quad (a+b)^2 = 9ab$

$\Rightarrow \quad \log(a+b)^2 = \log 9ab$

$\Rightarrow \quad 2\log(a+b) = 2\log 3 + \log a + \log b$

$\Rightarrow \quad 2(\log(a+b) - \log 3) = \log a + \log b$

$\Rightarrow \quad \log\left(\frac{a+b}{3}\right) = \frac{1}{2}(\log a + \log b)$

Example 7 If $n > 1$, then prove that $\frac{1}{\log_2 n} + \frac{1}{\log_3 n} + \ldots + \frac{1}{\log_{53} n} = \frac{1}{\log_{53!} n}$.

Sol. The given expression is equal to

$$\log_n 2 + \log_n 3 + \ldots + \log_n 53$$

$$= \log_n (2\cdot 3 \ldots 53) = \log_n 53! = \frac{1}{\log_{53!} n}$$

Example 8 Prove that $\dfrac{\log_a n}{\log_{ab} n} = 1 + \log_a b.$

Sol. $\dfrac{\log_a n}{\log_{ab} n} = \dfrac{\log_n ab}{\log_n a}$

$$= \frac{\log_n a + \log_n b}{\log_n a}$$

$$= 1 + \frac{\log_n b}{\log_n a} = 1 + \log_a b$$

Example 9 Compute $\log_{ab} (\sqrt[3]{a} / \sqrt{b})$ if $\log_{ab} a = 4$.

Sol. $\log_{ab} a = 4$

$$\Rightarrow \quad \frac{1}{\log_a ab} = 4$$

$$\Rightarrow \quad \frac{1}{\log_a a + \log_a b} = 4$$

$$\Rightarrow \quad 1 + \log_a b = \frac{1}{4}$$

$$\Rightarrow \quad \log_a b = -\frac{3}{4}$$

Now $\log_{ab} (\sqrt[3]{a} / \sqrt{b}) = \dfrac{\log(\sqrt[3]{a} / \sqrt{b})}{\log ab} = \dfrac{\frac{1}{3}\log a - \frac{1}{2}\log b}{\log a + \log b}$

$$= \frac{\frac{1}{3} - \frac{1}{2}\frac{\log b}{\log a}}{1 + \frac{\log b}{\log a}} = \frac{\frac{1}{3} - \frac{1}{2}\log_a b}{1 + \log_a b} = \frac{\frac{1}{3} + \frac{1}{2}\frac{3}{4}}{1 - \frac{3}{4}} = \frac{\frac{17}{24}}{\frac{1}{4}} = \frac{17}{6}$$

Example 10 Solve for x: $11^{4x-5} \cdot 3^{2x} = 5^{3-x} \cdot 7^{-x}$

Sol. $11^{4x-5} \cdot 3^{2x} = 5^{3-x} \cdot 7^{-x}$

$$\Rightarrow \quad (4x - 5)\log 11 + 2x\log 3 = (3 - x)\log 5 - x\log 7$$

$$\Rightarrow \quad x = \frac{\log(11^5 \cdot 5^3)}{\log(11^4 \cdot 315}$$

Example 11 Suppose that a and b are positive real numbers such that $\log_{27}a + \log_9 b = \frac{7}{2}$ and $\log_{27}b + \log_9 a = \frac{2}{3}$. Then find the value of ab.

Sol. $\log_{27}a + \log_9 b = \frac{7}{2}$ and $\log_{27}b + \log_9 a = \frac{2}{3}$

$$\Rightarrow \quad \frac{1}{3}\log_3 a + \frac{1}{2}\log_3 b = \frac{7}{2}$$

$$\text{and} \quad \frac{1}{3}\log_3 b + \frac{1}{2}\log_3 a = \frac{2}{3}$$

Adding these equations, we get

$$\Rightarrow \quad \frac{1}{3}\log_3(ab) + \frac{1}{2}\log_3(ab) = \frac{7}{2} + \frac{2}{3}$$

$$\Rightarrow \quad \frac{5}{6}\log_3(ab) = \frac{25}{6}$$

$$\Rightarrow \quad \log_3(ab) = 5$$

$$\Rightarrow \quad ab = 3^5 = 243$$

Example 12 Which of the following is greater; $m = (\log_2 5)^2$ or $n = \log_2 20$?

Sol. $m - n = (\log_2 5)^2 - [\log_2 5 + 2]$

let $\log_2 5 = x$

$$\Rightarrow \quad m - n = x^2 - x - 2 = (x-2)(x+1) = (\log_2 5 - 2)(\log_2 5 + 1) > 0$$

Hence, $m > n$

Example 13 If $\log_{12} 27 = a$, then prove that $\log_6 16 = 4\left(\frac{3-a}{3+a}\right)$.

Sol. $\because a = \log_{12} 27 = \log_{12}(3)^3 = 3\log_{12} 3$

$$= \frac{3}{\log_3 12} = \frac{3}{1+\log_3 4} = \frac{3}{1 + 2\log_3 2}$$

$$\therefore \quad \log_3 2 = \frac{3-a}{2a}$$

Then, $\log_6 16 = \log_6 2^4 = 4\log_6 2 = \dfrac{4}{\log_2 6}$

$$= \frac{4}{1+\log_2 3} = \frac{4}{1 + \frac{2a}{3-a}} = 4\left(\frac{3-a}{3+a}\right)$$

Example 14 Find the value of $6^{\log_{10}40} \cdot 5^{\log_{10}36}$.

Sol. We have $N = 6^{\log_{10}40} \cdot 5^{\log_{10}36}$

$$\therefore \quad \log_{10} N = \log_{10} 40 \cdot \log 6 + \log_{10} 36 \cdot \log 5$$

$$= \log_{10} 6\,[\log_{10} 40 + \log_{10} 25]$$
$$= \log_{10} 6\,[\log_{10} 1000]$$
$$= \log_{10} (6)^3$$
$$\therefore \quad N = 6^3 = 216$$

Example 15 If $y = a^{\frac{1}{1-\log_a x}}$ and $z = a^{\frac{1}{1-\log_a y}}$, then prove that $x = a^{\frac{1}{1-\log_a z}}$.

Sol. $\log_a y = \dfrac{1}{1-\log_a x}$

$$\therefore \quad 1 - \log_a y = 1 - \frac{1}{1-\log_a x} = \frac{-\log_a x}{1-\log_a x}$$

or
$$\frac{1}{1-\log_a y} = \frac{1-\log_a x}{-\log_a x} \qquad \text{(i)}$$

But
$$z = a^{\frac{1}{1-\log_a y}}$$

$$\Rightarrow \quad \log_a z = \frac{1}{1-\log_a y} = -\frac{1}{\log_a x} + 1$$

$$\Rightarrow \quad \frac{1}{\log_a x} = 1 - \log_a z$$

$$\Rightarrow \quad \log_a x = \frac{1}{1-\log_a z}$$

$$\therefore \quad x = a^{\frac{1}{1-\log_a z}}$$

Example 16 If $a \geq b > 1$, then find the largest possible value of the expression $\log_a(a/b) + \log_b(b/a)$.

Sol. Let $x = \log_a\left(\dfrac{a}{b}\right) + \log_b\left(\dfrac{b}{a}\right) = \log_a a - \log_a b + \log_b b - \log_b a = 2 - (\log_b a + \log_a b)$

Now x will be maximum if $\log_b a + \log_a b$ is minimum.

But $\log_b a + \log_a b = \left(\sqrt{\log_b a} - \sqrt{\log_a b}\right)^2 + 2$

Hence, its minimum value is 2.

$\therefore\ x_{max} = 2 - 2 = 0$

Example 17 If $\dfrac{\log_a N}{\log_c N} = \dfrac{\log_a N - \log_b N}{\log_b N - \log_c N}$, where $N > 0$ and $N \neq 1$, $a, b, c > 0$ and not equal to 1, then prove that $b^2 = ac$.

Sol. $\dfrac{\log_a N}{\log_c N} = \dfrac{\log_a N - \log_b N}{\log_b N - \log_c N}$

$$\Rightarrow \quad \frac{\log_N c}{\log_N a} = \frac{\dfrac{1}{\log_N a} - \dfrac{1}{\log_N b}}{\dfrac{1}{\log_N b} - \dfrac{1}{\log_N c}}$$

$$\Rightarrow \quad \frac{\log_N c}{\log_N a} = \frac{\log_N c}{\log_N a} \times \frac{\log_N b - \log_N a}{\log_N c - \log_N b}$$

$$\Rightarrow \quad \frac{\log_N b - \log_N a}{\log_N c - \log_N b} = 1$$

$$\Rightarrow \quad \log_N b - \log_N a = \log_N c - \log_N b$$

$$\Rightarrow \quad b/a = c/b$$

$$\Rightarrow \quad b^2 = ac$$

Example 18 Find the value of $\dfrac{\log_2 24}{\log_{96} 2} - \dfrac{\log_2 192}{\log_{12} 2}$.

Sol. Let $\log_2 12 = a$. Then

$$\frac{1}{\log_{96} 2} = \log_2 96 = \log_2 2^3 \times 12 = 3 + a;$$

$$\log_2 24 = 1 + a$$

$$\log_2 192 = \log_2 (16 \times 12) = 4 + a \text{ and } \frac{1}{\log_{12} 2} = \log_2 12 = a.$$

Therefore, the given expression = $(1 + a)(3 + a) - (4 + a)a = 3$

Example 19 If $2^{x+y} = 6^y$ and $3^{x-1} = 2^{y+1}$, then find the value of $(\log 3 - \log 2)/(x - y)$.

Sol. Taking log of both sides of $2^{x+y} = 6^y$, we have

$$(x + y) \log 2 = y (\log 2 + \log 3)$$

or $\quad x \log 2 = y \log 3$

or $$\frac{x}{\log 3} = \frac{y}{\log 2} = \frac{x - y}{\log 3 - \log 2} = \lambda \text{ (say)} \qquad (1)$$

Also, $(x - 1) \log 3 = (y + 1) \log 2$ (From 2nd eq).

or $x \log 3 - y \log 2 = \log 3 + \log 2$

$$\Rightarrow \quad \lambda [(\log 3)^2 - (\log 2)^2] = \log 3 + \log 2 \qquad \text{[Using(1)]}$$

$$\therefore \quad \lambda = \frac{1}{\log 3 - \log 2} \text{ or } \frac{1}{\lambda} = \log 3 - \log 2 = \log \frac{3}{2}$$

Example 20 If $\log_2 x + \log_2 y \geq 6$, then find the least value of $x + y$.

Sol. Given $\log_2 x + \log_2 y \geq 6$

$\Rightarrow \quad \log_2 (xy) \geq 6$

$\Rightarrow \quad xy \geq 64$

Also for $\log_2 x$ and $\log_2 y$ to be defined

$x > 0, y > 0$

Since A.M. ≥ G.M.

$$\therefore \quad \frac{x+y}{2} \geq \sqrt{xy}$$

$$\Rightarrow \quad x + y \geq 2\sqrt{xy} \geq 2\sqrt{64} = 16$$

Example 21 If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, then prove that $f(x_1) + f(x_2) = f\left(\frac{x_1+x_2}{1+x_1x_2}\right)$.

Sol. $f(x_1) + f(x_2) = \log\left(\frac{1+x_1}{1-x_1}\cdot\frac{1+x_2}{1-x_2}\right)$

$$= \log\left(\frac{1+x_1x_2+x_1+x_2}{1+x_1x_2-x_1+x_2}\right)$$

$$= \log\left(\frac{1+\dfrac{x_1+x_2}{1+x_1x_2}}{1-\dfrac{x_1+x_2}{1+x_1x_2}}\right) = f\left(\frac{x_1+x_2}{1+x_1x_2}\right)$$

Example 22 If $\log_2 10 = p$; $\frac{\log_e 10}{\log_e 7} = q$ and $(11)^r = 10$ then prove that $\log_{10}154 = \frac{pq+qr+rp}{pqr}$.

Sol. $\log_{10}154 = \log_{10}2 + \log_{10}7 + \log_{10}11$

Now $\log_2 10 = p$

$$\Rightarrow \quad \frac{1}{\log_{10}2} = p$$

$$\Rightarrow \quad \log_{10}2 = \frac{1}{p} \qquad (1)$$

$$\text{Also,} \quad \frac{\log_e 10}{\log_e 7} = q$$

$$\Rightarrow \quad \frac{1}{\log_{10}7} = q$$

$$\Rightarrow \quad \log_{10}7 = \frac{1}{q} \qquad (2)$$

And $11^r = 10$

$\Rightarrow \quad r\log_{10}11 = 1$

$$\Rightarrow \quad \log_{10}11 = \frac{1}{r} \tag{3}$$

$$\therefore \quad \log_{10}154 = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{pq+qr+rp}{pqr}$$

Example 23 Prove that $\log_a N \cdot \log_b N + \log_b N \cdot \log_c N + \log_c N \cdot \log_a N = \dfrac{\log_a N \cdot \log_b N \cdot \log_c N}{\log_{abc} N}$

Sol. $\log_a N.\ \log_b N + \log_b N.\ \log_c N + \log_c N.\ \log_a N$

$$= \frac{1}{\log_N a \log_N b} + \frac{1}{\log_N b \log_N c} + \frac{1}{\log_N c \log_N a}$$

$$= \frac{\log_N a + \log_N b + \log_N c}{(\log_N a)(\log_N b)(\log_N c)}$$

$$= \frac{\log_N abc}{\dfrac{1}{\log_a N}\dfrac{1}{\log_b N}\dfrac{1}{\log_c N}}$$

$$= \frac{\log_a N.\log_b N.\log_c N}{\log_{abc} N}$$

Law of Change of Base

Law 6: For $m, a, b > 0$ and $a \neq 1, b \neq 1$, $\log_a m = \dfrac{\log_b m}{\log_b a}$

Proof: Let $\log_a m = x$. Then , $a^x = m$.

$\Rightarrow \quad \log_b(a^x) = \log_b m$ [Taking log to the base b]

$\Rightarrow \quad x \log_b a = \log_b m$

$\Rightarrow \quad \log_a m.\ \log_b a = \log_b m$ [$\because \log_m m = 1$]

$$\Rightarrow \quad \log_a m = \frac{\log_b m}{\log_b a}$$

Replacing b by m in the above result, we get

$$\log_a m = \frac{\log_m m}{\log_m a}$$

$$\Rightarrow \quad \log_a m = \frac{1}{\log_m a} \quad [\because \log_m m = 1]$$

Example 24 Find the value of $\log_3 4 \log_4 5 \log_5 6 \log_6 7 \log_7 8 \log_8 9$

Sol. $\log_3 4 \log_4 5 \log_5 6 \log_6 7 \log_7 8 \log_8 9$

$$= \frac{\log 4}{\log 3} \times \frac{\log 5}{\log 4} \times \frac{\log 6}{\log 5} \times \frac{\log 7}{\log 6} \times \frac{\log 8}{\log 7} \times \frac{\log 9}{\log 8}$$

$$= \frac{\log 9}{\log 3} = \log_3 9 = 2$$

Example 25 If $y^2 = xz$ and $a^x = b^y = c^z$, then prove that $\log_b a = \log_c b$.

Sol. $a^x = b^y = c^z$

$\Rightarrow \quad x \log a = y \log b = z \log c$

Now, $\quad \frac{y}{x} = \frac{z}{y}$

$\Rightarrow \quad \frac{\log a}{\log b} = \frac{\log b}{\log c}$

$\Rightarrow \quad \log_b a = \log_c b$

Example 26 If $\log_b a \cdot \log_c a + \log_a b \cdot \log_c b + \log_a c \cdot \log_b c = 3$, then prove that $abc = 1$.

Sol. $\log_b a \cdot \log_c a + \log_a b \cdot \log_c b + \log_a c \cdot \log_b c = 3$

$\Rightarrow \quad \frac{\log a}{\log b}\frac{\log a}{\log c} + \frac{\log b}{\log a}\frac{\log b}{\log c} + \frac{\log c}{\log a}\frac{\log c}{\log b} = 3$

$\Rightarrow \quad (\log a)^3 + (\log b)^3 + (\log c)^3 = (\log a)(\log b)(\log c)$

$\Rightarrow \quad \log a + \log b + \log c = 0$ (as a, b, c are different)

$\Rightarrow \quad \log abc = 0$

$\Rightarrow \quad abc = 1$

Example 27 If $\log_a 3 = 2$ and $\log_b 8 = 3$, then find the value of $\log_a b$.

Sol. $\log_b 8 = 3 \Rightarrow 3 \log_b 2 = 3 \Rightarrow \log_b 2 = 1$

Now, $\log_a b = \log_2 b \cdot \log_a 2 = \log_2 b \cdot \log_3 2 \cdot \log_a 3$

$= 1 \cdot \log_3 2 \cdot 2 = 2 \log_3 2 = \log_3 4$

Example 28 If $x = \log_{2a} a$, $y = \log_{3a} 2a$ and $z = \log_{4a} 3a$, then prove that $1 + xyz = 2yz$.

Sol. $1 + xyz = 1 + (\log_{2a} a)(\log_{3a} 2a)(\log_{4a} 3a)$

$$= 1 + \frac{\log a}{\log 2a}\frac{\log 2a}{\log 3a}\frac{\log 3a}{\log 4a}$$

$$= 1 + \frac{\log a}{\log 4a}$$

$= \log_{4a}4a + \log_{4a}a$

$= \log_{4a}4a^2$

$= 2\log_{4a}2a$

$= 2(\log_{3a} 2a)(\log_{4a} 3a) = 2yz$

Example 29 If $a^x = b^y = c^z = d^w$, show that $\log_a(bcd) = x\left(\frac{1}{y}+\frac{1}{z}+\frac{1}{w}\right)$.

Sol. $\log_a(bcd) = \log_a b + \log_a c + \log_a d$

Now, $a^x = b^y \Rightarrow x\log a = y\log b \Rightarrow \frac{\log b}{\log a} = \frac{x}{y} \Rightarrow \log_a b = \frac{x}{y}$

Similarly, $\log_a c = \frac{x}{z}$ and $\log_a d = \frac{x}{w}$.

$$\Rightarrow \quad \log_a(bcd) = \log_a b + \log_a c + \log_a d = x\left(\frac{1}{y}+\frac{1}{z}+\frac{1}{w}\right)$$

Example 30 If $a = \log_{12} 18$, $b = \log_{24} 54$ then find the value of $ab + 5(a-b)$.

Sol. We have $a = \log_{12} 18 = \frac{\log_2 18}{\log_2 12} = \frac{1+2\log_2 3}{2+\log_2 3}$

and $\quad b = \log_{24} 54 = \frac{\log_2 54}{\log_2 24} = \frac{1+3\log_2 3}{3+\log_2 3}$

Putting $x = \log_2 3$, we have

$$ab + 5(a-b) = \frac{1+2x}{2+x}\cdot\frac{1+3x}{3+x} + 5\left(\frac{1+2x}{2+x} - \frac{1+3x}{3+x}\right)$$

$$= \frac{6x^2+5x+1+5(-x^2+1)}{(x+2)(x+3)}$$

$$= \frac{x^2+5x+6}{(x+2)(x+3)} = 1$$

Law 7: For $a, n > 0$ and $a \neq 1$; $a^{\log_a n} = n$.

Proof: Let $\log_a n = x$.

Then $a^x = n$.

$\Rightarrow \quad a^{\log_a n} = n$ [Putting the value of x in $a^x = n$]

e.g, $3^{\log_3 8} = 8$, $2^{3\log_2 5} = 2^{\log_2 5^3} = 5^3$, $5^{-2\log_5 3} = 5^{\log_5 3^{-2}} = 3^{-2} = \frac{1}{9}$

Law 8: $a^{\log_b c} = c^{\log_b a}$

Proof: Let $a^{\log_b c} = p$

$\Rightarrow \quad \log_b c = \log_a p$

$\Rightarrow \quad \frac{\log c}{\log b} = \frac{\log p}{\log a}$

$\Rightarrow \quad \frac{\log a}{\log b} = \frac{\log p}{\log c}$

$\Rightarrow \quad \log_b a = \log_c p$

$$\Rightarrow \quad p = c^{\log_b a}$$

$$\Rightarrow \quad a^{\log_b c} = c^{\log_b a}$$

Eample 31 Find the value of $81^{(1/\log_5 3)} + 27^{\log_9 36} + 3^{4/\log_7 9}$.

Sol. $81^{(1/\log_5 3)} + 27^{\log_9 36} + 3^{4/\log_7 9}$

$$= \left(3^4\right)^{\log_3 5} + (3^3)^{\log_{3^2}(6^2)} + 3^{4\log_9 7}$$

$$= 3^{\log_3 5^4} + (3^3)^{\log_3(6)} + 3^{4\log_{3^2} 7}$$

$$= 5^4 + 3^{\log_3 6^3} + 3^{2\log_3 7}$$

$$= 5^4 + 6^3 + 3^{\log_3 7^2}$$

$$= 625 + 216 + 7^2 = 890$$

Example 32 If $(4)^{\log_9 3} + (9)^{\log_2 4} = (10)^{\log_x 83}$, then find the value of x.

Sol. $(4)^{\log_9 3} + (9)^{\log_2 4} = (10)^{\log_x 83}$

$$\Rightarrow \quad (4)^{\frac{1}{2}\log_3 3} + (9)^{2\log_2 2} = (10)^{\log_x 83}$$

$$\Rightarrow \quad 2 + 81 = (10)^{\log_x 83}$$

$$\Rightarrow \quad 83 = (10)^{\log_x 83}$$

$$\Rightarrow \quad x = 10$$

Example 33 Find the value of $49^{(1-\log_7 2)} + 5^{-\log_5 4}$.

Sol. $49^{(1-\log_7 2)} + 5^{-\log_5 4}$

$$= 49 \times 7^{-2\log_7 2} + 5^{-\log_5 4}$$

$$= 49 \times \frac{1}{4} + \frac{1}{4} = \frac{50}{4} = \frac{25}{2}$$

Example 34 Find the value of $3^{\log_4 5} - 5^{\log_4 3}$.

Sol. Let $3^{\log_4 5} = a$

$$\Rightarrow \quad \log_4 5 = \log_3 a$$

$$\Rightarrow \quad \frac{\log 5}{\log 4} = \frac{\log a}{\log 3}$$

$$\Rightarrow \quad \frac{\log a}{\log 5} = \frac{\log 3}{\log 4}$$

$$\Rightarrow \quad \log_5 a = \log_4 3$$

$$\Rightarrow \quad a = 5^{\log_4 3}$$

$$\Rightarrow \quad 3^{\log_4 5} - 5^{\log_4 3} = 0$$

Example 35 If $2x^{\log_4 3} + 3^{\log_4 x} = 27$, then find the value of x.

Sol. $2\,x^{\log_4 3} + 3^{\log_4 x} = 27$

$$\Rightarrow \quad 2.3^{\log_4 x} + 3^{\log_4 x} = 27$$

$$\Rightarrow \quad 3^{\log_4 x} = 9 = 3^2$$

$$\Rightarrow \quad \log_4 x = 2$$

$$\therefore \quad x = 4^2 = 16$$

Example 36 Find the value of $(2^{\log_6 18}).(3^{\log_6 3})$

Sol. $(2^{\log_6 18}).(3^{\log_6 3})$

$$= (2^{\log_6 6 + \log_6 3}).(3^{\log_6 3})$$

$$= 2(2^{\log_6 3}).(3^{\log_6 3})$$

$$= 2 \cdot 6^{\log_6 3}$$

$$= 2 \times 3 = 6$$

Example 37 If $3^{(\log_3 7)^x} = 7^{(\log_7 3)^x}$, then find the value of x.

Sol. Consider $a^{\sqrt{\log_a b}} = b^{\sqrt{\log_b a}}$

$$\therefore \quad \sqrt{\log_a b} = \sqrt{\log_b a} \cdot \log_a b \qquad \text{(taking log with base '}a\text{')}$$

$$\Rightarrow \quad 1 = \sqrt{\log_b a} \cdot \sqrt{\log_a b} \text{ which is true}$$

Thus, from $3^{(\log_3 7)^x} = 7^{(\log_7 3)^x}$, we have

$$x = \frac{1}{2}$$

Example 38 Find the value of $\dfrac{81^{\frac{1}{\log_5 9}} + 3^{\frac{3}{\log_{\sqrt{6}} 3}}}{409}\left((\sqrt{7})^{\frac{2}{\log_{25} 7}} - (125)^{\log_{25} 6}\right)$.

Sol. $\dfrac{81^{\frac{1}{\log_5 9}} + 3^{\frac{3}{\log_{\sqrt{6}} 3}}}{409}\left((\sqrt{7})^{\frac{2}{\log_{25} 7}} - (125)^{\log_{25} 6}\right)$

$$= \frac{9^{\log_9 5^2} + 3^{\log_3 (\sqrt{6})^3}}{409}\left(7^{\log_7 25} - 5^{\log_5 6^{3/2}}\right)$$

$$= \frac{25 + 6\sqrt{6}}{409}[25 - 6\sqrt{6}]$$

$$= \frac{625 - 216}{409} = \frac{409}{409} = 1$$

Exercise 1

1. Evaluate each of the following:
   (i) $\log_{10}5 + \log_{10}2$
   (ii) $\frac{1}{2}\log_{10}36 + \log_{10}5 - \log_{10}30$
   (iii) $\log_{10}5 + 2\log_{10}0.5 + 3\log_{10}2$
2. Prove that $\log\frac{11}{5} + \log\frac{14}{3} - \log\frac{22}{15} = \log 7$.
3. Prove that $\log\frac{70}{33} + \log\frac{22}{135} - \log\frac{7}{18} = 3\log 2 - 2\log 3$.
4. Find the values of the following:
   (i) $-\log_5 \log_3 \sqrt{\sqrt[5]{9}}$ (ii) $\frac{1}{6}\log_{\frac{\sqrt{3}}{2}}\left(\frac{64}{27}\right)$
5. Find the values of $3^{2\log_9 3}$.
6. Find the smallest integer greater than $\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi}$.
7. If $\log_{10} x = y$, then find $\log_{1000} x^2$ in terms of y.
8. What is logarithm of $32\sqrt[5]{4}$ to the base $2\sqrt{2}$?
9. If $\log_a(ab) = x$ then evaluate $\log_b(ab)$ in terms of x.
10. Which is greater, $x = \log_3 5$ or $y = \log_{17} 25$?
11. If $y = \frac{1}{2^{\log_x 4}}$, then find x in terms of y.
12. If $\log_e\left(\frac{a+b}{2}\right) = \frac{1}{2}(\log_e a + \log_e b)$, then find the relation between a and b.
13. If a, b, c are consecutive positive integers and $\log(1 + ac) = 2K$ then find the value of K.
14. If $\log_k x \cdot \log_5 k = \log_x 5$, $k \neq 1$, $k > 0$, then find the value of x.
15. Let $3^a = 4, 4^b = 5, 5^c = 6, 6^d = 7, 7^e = 8$ and $8^f = 9$. Then the value of the product $(abcdef)$.
16. Find the value of $\frac{1 + 2\log_3 2}{(1 + \log_3 2)^2} + (\log_6 2)^2$.
17. Find the value of $\log_3 5 \cdot \log_{25} 27$.
18. Find the greatest integer less than or equal to the number $\log_2 15 \times \log_{1/6} 2 \times \log_3 1/6$.

19. Find the value of $2^{\log_{(2\sqrt{2})} 15}$.

20. Find the value of $4^{5\log_{4\sqrt{2}}(3-\sqrt{6})-6\log_8(\sqrt{3}-\sqrt{2})}$.

21. Find the value of $\left(\frac{1}{49}\right)^{1+\log_7 2}+5^{-\log_{(1/5)}(7)}$.

Exercise 2

1. Prove that $\log_{ab} x=\dfrac{\log_a x \log_b x}{\log_a x+\log_b x}$.

2. If 'x' and 'y' are real numbers such that $2\log(2y-3x)=\log x+\log y$, then find $\dfrac{x}{y}$.

3. If $a^4 \cdot b^5=1$ then find the value of $\log_a(a^5b^4)$.

4. If $\log_a b=2$; $\log_b c=2$ and $\log_3 c=3+\log_3 a$ then find the value of $\dfrac{c}{ab}$.

5. If $a^x=b$, $b^y=c$, $c^z=a$, then find the value xyz.

6. Simplify $\dfrac{1}{1+\log_a bc}+\dfrac{1}{1+\log_b ca}+\dfrac{1}{1+\log_c ab}$.

7. Find the value of $\dfrac{1}{\log_{bc} abc}+\dfrac{1}{\log_{ca} abc}+\dfrac{1}{\log_{ab} abc}$ $(a, b, c>0)$.

8. Suppose that $x, y, z>0$ and different from 1 and $\log x+\log y+\log z=0$. Find the value of $x^{\frac{1}{\log y}+\frac{1}{\log z}} \cdot y^{\frac{1}{\log z}+\frac{1}{\log x}} \cdot z^{\frac{1}{\log x}+\frac{1}{\log y}}$ (base 10).

9. Find the value of $\dfrac{\log_5 250}{\log_{50} 5}-\dfrac{\log_5 10}{\log_{1250} 5}$.

10. Given that $\log_2 3=a, \log_3 5=b, \log_7 2=c$, then prove that $\log_{140} 63=\dfrac{1+2ac}{2c+1+abc}$.

11. Find the value of $\sqrt[3]{\left(5^{1/\log_7 5}+\dfrac{1}{\sqrt{(-\log_{10} 0.1)}}\right)}$.

12. If there exist positive integers A, B and C with no common factors greater than 1, such that $A\log_{200}5+B\log_{200}2=C$, then find the sum $A+B+C$.

13. Find the value of x satisfying the equations $\log_3(\log_2 x)+\log_{1/3}(\log_{1/2} y)=1$ and $xy^2=9$.

14. Let a and b be real numbers greater than 1 for which there exists a positive real number c, different from 1, such that $2(\log_a c+\log_b c)=9\log_{ab} c$. Then find the largest possible value of $\log_a b$.

CHAPTER

3

Exponential and Logarithmic Functions

EXPONENTIAL FUNCTION

If a is any number such that $a > 0$ and $a \neq 1$ then an exponential function is a function in the form $f(x) = a^x$, where a is called the base and x can be any real number. e.g., $f(x) = 2^x$, $g(x) = (4/7)^x$ are exponential functions.

There is a big difference between an exponential function and a polynomial. The function $g(x) = x^3$ is a polynomial. Here, the variable, x, is being raised to some constant power. The function $f(x)=3^x$ is an exponential function; the variable is the exponent.

Here, the restriction on the base a is that it is not 0 or 1 as $f(x) = 0^x = 0$ and $f(x) = 1^x = 1$ and hence, these are constant functions and won't have many of the same properties that general exponential functions have. Also, we avoid negative numbers as base so that we don't get any complex values out of the function evaluation. For instance, if we allowed $a = -4$ the function would be, $f(x) = (-4)^x$, then we have $f(0.5) = (-4)^{0.5}$ which is a complex number. We only want real numbers to arise from function evaluation and so to make sure of this we require that a is not a negative number.

Graph of Exponential Function and its Properties

Consider exponential function $y = f(x) = 2^x$. Here $f(0) = 1$, and on increasing the value of x from $x = 0$ onwards value of y increases, i.e., $f(1) = 2, f(2) = 4, f(10) = 1024$ .... When x approaches to infinity, 2^x approaches to infinity. Thus $f(x)$ is an increasing function. Now, consider some negative values of x, i.e., $f(-1) = 0.5, f(-2) = 0.25, f(-3) = 0.125$ .... Thus, graph of the function gets closer to x-axis, but never touches or crosses it, as $2^x > 0, \forall x \in R$. When x approaches to negative infinity, 2^x approaches to 0. This means that there is a horizontal asymptote at the x-axis or $y = 0$.

From the above discussion, the graph of $y = f(x) = 2^x$ can be plotted as shown here.

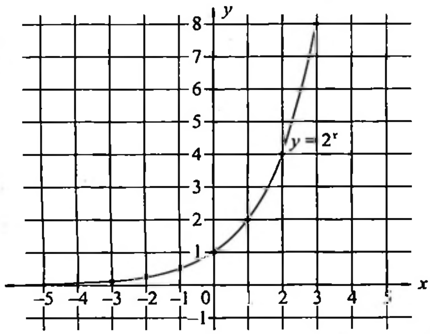

Fig. 3.1

From the graph, we can see that $f(x) = 2^x$ is an increasing function in its domain R. Also, the graph always remains above x-axis, i.e., $2^x > 0$. Thus, range of the function is $(0, \infty)$. Similarly, we can draw the graph of $y = f(x) = 3^x$ passing through the point (0, 1) and having the same nature as the graph of $f(x) = 2^x$. In fact, this pattern of the graph is common for $f(x) = a^x$ for any base which is greater than 1 ($a > 1$). Also, all graphs pass through the point (0, 1) as $a^0 = 1$ for any value of a. For $a > 1$, when $x_2 > x_1$, we have $a^{x_2} > a^{x_1}$

Now, consider the function $y = f(x) = (0.5)^x$. Here, $f(0) = 1$, and on increasing the value of x from $x = 0$ onwards value of y decreases as $f(1) = 0.5, f(2) = 0.25, \ldots$. When x approaches to infinity, 2^x approaches to zero but never becomes exactly zero. Thus, $f(x)$ is decreasing function. Also, $f(-1) = 2, f(-2) = 4, f(-3) = 8$ and so on. Thus, value of 2^x increases as the value of x becomes more and more negative. The graph of the function can be plotted as shown here.

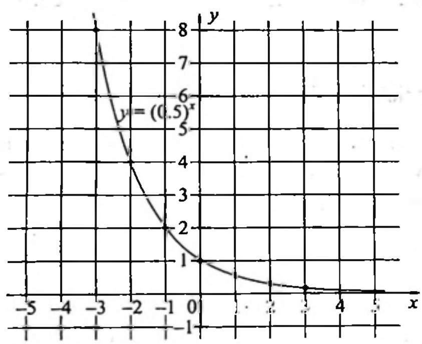

Fig. 3.2

From the graph, we can see that $y = f(x) = (0.5)^x$ is an decreasing function in its domain. In fact, this pattern of the graph is common for $f(x) = a^x$ for any base which

is less than 1(0 < a < 1). Also, all graphs pass through the point (0, 1) as $a^0 = 1$ for any value of a. For $a > 1$, when $x_2 > x_1$, we have $a^{x_1} > a^{x_2}$

Graph of Exponential Function for Different Bases

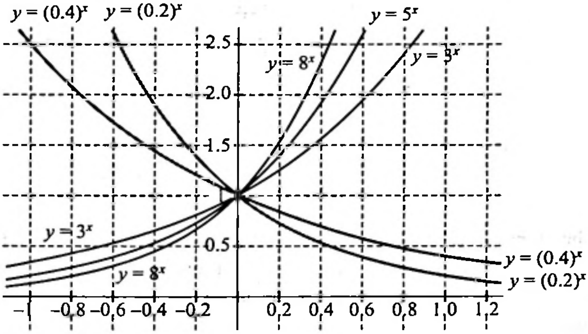

Fig. 3.3

LOGARITHMIC FUNCTION

The **logarithm** of a number to a given base is the exponent to which the base must be raised in order to produce that number. For example, the logarithm of 1000 to base 10 is 3, because 10 to the power of 3 is 1000, i.e., $10^3 = 1000$. We write $\log_{10}1000 = 3$. Here '10' is called the base of the logarithm. Similarly $\log_2 64$ is the value to which '2' must be raised to get 64. Since $2^6 = 64$, $\log_2 64 = 6$. Thus, if $a^x = y$, then we have $\log_a y = x$. In exponential function $y = a^x$, x can take any real value, so $\log_a y$ can also take any real value. Also, $y = a^x > 0$, so $\log_a y$ is defined only if $y > 0$. Thus, we can define logarithmic function as $f(x) = \log_a x$, $a > 0$, $a \neq 1$, having domain $(0, \infty)$ and range R. Thus, logarithmic function is actually inverse of exponential function. So, domain and range of exponential function are range and domain of logarithmic function, respectively.

Graphs of the Logarithmic Function

If we consider any point (p, q) on the graph of $y = a^x$, we have $q = a^p$. So, we have $p = \log_a q$, or there is a point (q, p) on the graph of $y = \log_a x$. Thus, for any point (p, q) on the graph of $y = a^x$, we have a point (q, p) on the graph of $y = \log_a x$. Since point (q, p) is the mirror image of the point (p, q) in the line $y = x$, we can draw the graph of $y = \log_a x$ by taking the mirror image of the graph of $y = a^x$ in line $y = x$. Thus, graphs of $y = a^x$ and $y = \log_a x$ are symmetrical about the line $y = x$ as shown in the following figure.

Case I: When $a > 1$

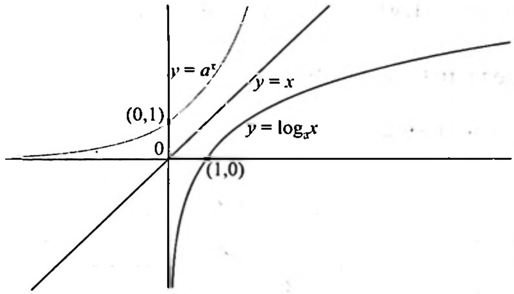

Fig. 3.4

From the graph, we can see that $y = \log_a x$ is an increasing function in its domain. The value of $\log_a x$ approaches to infinity as x approaches to infinity. The value of $\log_a x$ approaches to negative infinity as x approaches to 0. Also, for $x_2 > x_1$, we have $\log_a x_2 > \log_a x_1$.

Case II: When $0 < a < 1$

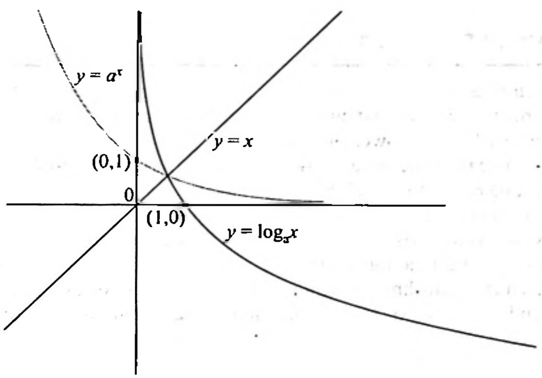

Fig. 3.5

From the graph, we can see that $y = \log_a x$ is decreasing function in its domain. The value of $\log_a x$ approaches to negative infinity as x approaches to infinity. The value of $\log_a x$ approaches to positive infinity as x approaches to 0. Also, for $x_2 > x_1$, we have $\log_a x_2 < \log_a x_1$.

Graphs of the Logarithmic Function for Different Bases

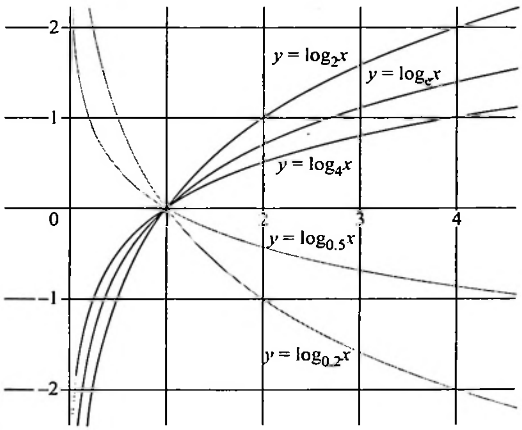

Fig. 3.6

Since $a^0 = 1$, we have $\log_a 1 = 0$. Thus, graph of $y = \log_a x$ for any value of base 'a' passes through the point (1, 0) on x-axis.

Also, since $a^1 = a$, we have $\log_a a = 1$, for any base 'a'.

Note:
- **Common logarithm** is the logarithm with base 10. It is also known as the **decadic logarithm**, named after its base. It is indicated by log10(x).
- **Natural logarithm** is the logarithm to the base e, where e is an irrational constant approximately equal to 2.718281828. Here, e is defined exactly as $e = (1 + 1/m)^m$ as m increases to infinity. The natural logarithm is generally written as ln(x) or $\log_e(x)$.

Example 1 Find the number of solutions of equation $(2x - 3)2^x = 1$.

Sol. We have $(2x - 3)2^x = 1$

or $\quad 2x - 3 = 2^{-x}$

To find the number of roots of the above equation, we need to find the number of points of intersection of $y = 2x - 3$ and $y = 2^{-x}$.

The graphs of these functions are as shown in the following figure.

From the figure, graphs intersect at only one point.

Hence, there is only one solution of the given equation.

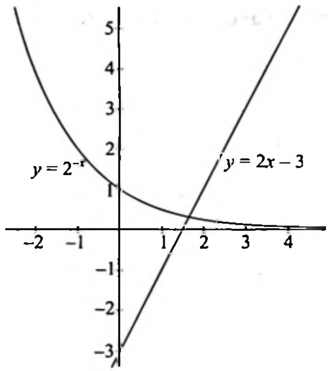

Fig. 3.7

Example 2 Find the number of solutions of $2^x + 3^x + 4^x - 5^x = 0$.

Sol. $2^x + 3^x + 4^x - 5^x = 0$

$$\Rightarrow \quad 2^x + 3^x + 4^x = 5^x$$

$$\Rightarrow \quad \left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x = 1$$

Now number of solutions of the equation is equal to the number of times

$$y = \left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x \text{ and } y = 1 \text{ intersect.}$$

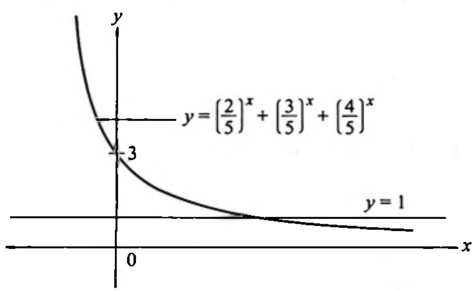

Fig. 3.8

From the graph, equation has only one solution.

Example 3 Find number of solutions of $|x|3^{|x|} = 1$.

Sol. We have $|x|3^{|x|} = 1$

or $\quad |x| = 3^{-|x|}$

Now $\quad 3^{-|x|} = \begin{cases} 3^{-x}, & x \geq 0 \\ 3^{x}, & x < 0 \end{cases}$

To find the number of roots of the above equation we need to find the number of points of intersection of $y = |x|$ and $y = 3^{-|x|}$.

The graphs of these functions are as shown in the following figure.

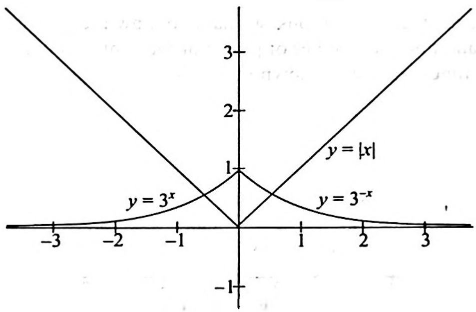

Fig. 3.9

From the graph number of solutions is 2.

Example 4 Find the number of solution to equation $\log_2 (x + 5) = 6 - x$:

Sol. Here, $x + 5 = 2^{6-x}$.

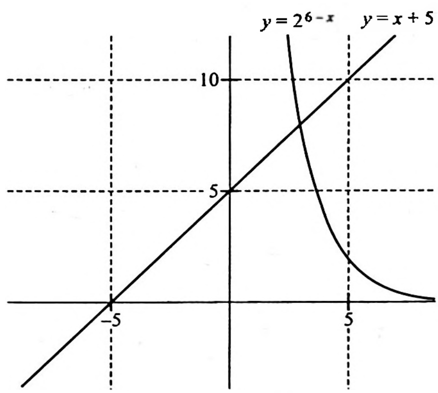

Fig. 3.10

Now graph of $y = x + 5$ and $y = 2^{6-x}$ intersect only once.
Hence, there is only one solution.

Example 5 Find the number of solutions of the following equations:

i. $x^{-1/2}\log_{0.5} x = 1$ **ii.** $x^2 - 4x + 3 - \log_2 x = 0$

Sol.

(i) We have, $x^{-1/2}\log_{0.5} x = 1$

$\Rightarrow \quad \log_{0.5} x = \sqrt{x}$

To find the number of solutions, we have to draw the graphs of $y = \sqrt{x}$ and $y = \log_{0.5} x$ and find the number of points of their intersection.
Graphs of functions are as shown in the following figure.

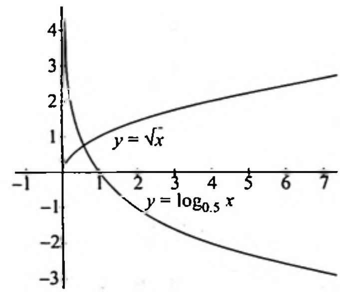

Fig. 3.11

From the figure, we see that graphs intersect at only one point.
Hence, there is only one solution.

(ii) We have $x^2 - 4x + 3 - \log_2 x = 0$
or $x^2 - 4x + 3 = \log_2 x$
Let us draw the graphs of $y = x^2 - 4x + 3$ and $y = \log_2 x$.

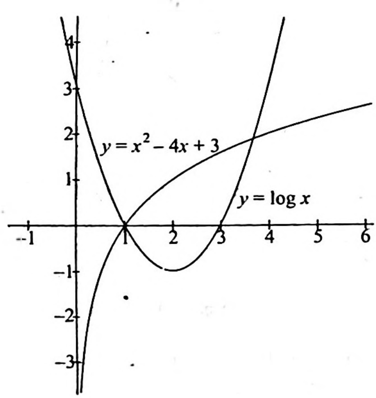

Fig. 3.12

From the figure, we see that graphs intersect at two points.
Hence, there are two solutions.

Example 6 Find number of roots of the equation $x^3 - \log_{0.5} x = 0$.

Sol. We have $x^3 - \log_{0.5} x = 0$

or $\quad x^3 = \log_{0.5} x$

To find the number of roots of the equation, we have to draw the graphs of $y = x^3$ and $y = \log_{0.5} x$.

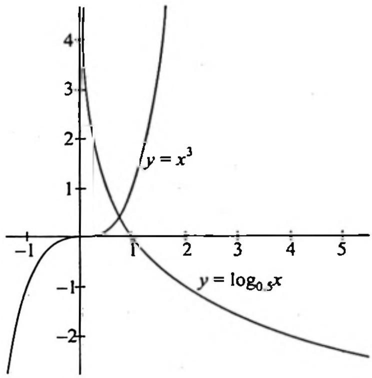

Fig. 3.13

From the figure, graphs intersect at only one point.

Hence, the equation has only one root.

Example 7 Which of the following pairs of expressions are defined for the same set of values of x?

(i) $f_1(x) = 2\log_{10} x$ and $f_2(x) = \log_{10} x^2$

(ii) $f_1(x) = \log_x x^2$ and $f_2(x) = 2$

(iii) $f_1(x) = \log_{10}(x-2) + \log_{10}(x-3)$ and $f_2(x) = \log_{10}(x-2)(x-3)$

Sol. (i) $f_1(x) = 2\log_{10} x$ is defined for $x > 0$

$f_2(x) = \log_{10} x^2$ is defined for $x^2 > 0$ or $x \in R - \{0\}$

Therefore $f_1(x)$ and $f_2(x)$ are not defined for same set of values of x.

(ii) $f_1(x) = \log_x x^2$ is defined for $x > 0, x \neq 1$

$\therefore \quad f_1(x) = 2, x > 0, x \neq 1$

But $f_2(x) = 2$ is defined for all real x.

Therefore $f_1(x)$ and $f_2(x)$ are not defined for same set of values of x.

(iii) $f_1(x) = \log_{10}(x-2) + \log_{10}(x-3)$ is defined

if $x - 2 > 0$ and $x - 3 > 0$

$\therefore \qquad x > 3$

$f_2(x) = \log_0(x-2)(x-3)$ is defined

if $(x-2)(x-3) > 0$

$\therefore \qquad x < 2$ or $x > 3$

Therefore $f_1(x)$ and $f_2(x)$ are not defined for same set of values of x.

CHAPTER

4

Logarithmic Equations

Equation involving logarithmic functions as terms is called logarithmic equation. While solving logarithmic equations, we tend to simplify the equation using laws of logarithm. Solving equation after simplification may give some roots which are not defining some of the terms in the initial equation. Thus while solving equations involving logarithmic functions, we must take care of domain of the original equation.

Example 1 Solve: $\log_9(4-x) = 0.5$

Sol. $\log_9(4-x) = 0.5$

$\Rightarrow \quad 4 - x = 9^{0.5}$

$\Rightarrow \quad 4 - x = 3$

$\Rightarrow \quad x = 1$

Example 2 Solve: $\log_{10} x - 1 = -\log_{10}(x-9)$

Sol. $\log_{10}x - 1 = -\log_{10}(x-9)$

$\Rightarrow \quad \log_{10}x + \log_{10}(x-9) = 1$

$\Rightarrow \quad lox_{10}x(x-9) = 1$

$\Rightarrow \quad x(x-9) = 10$

$\Rightarrow \quad x^2 - 9x - 10 = 0$

$\Rightarrow \quad (x-10)(x+1) = 0$

$\Rightarrow \quad x = 10, -1$

But for $x = -1$, $\log_{10}x$ in original equation is not defined.

Hence, $x = 10$ is the only solution.

Example 3 Solve: $\log_8 x + \log_8(x+6) = \log_8(5x+12)$

Sol. $\log_8 x + \log_8(x+6) = \log_8(5x+12)$

$\Rightarrow \quad \log_8 x(x+6) = \log_8(5x+12)$

$\Rightarrow \quad x^2 + 6x = 5x + 12$

$\Rightarrow \quad x^2 + x - 12 = 0$

$\Rightarrow \quad (x-3)(x+4) = 0$

$\Rightarrow \quad x = 3, -4$

But for $x = -4$, $\log_8 x$ term in original equation is not defined.

Hence, $x = 3$ is the only solution.

Example 4 Solve: $\log_4 8 + \log_4(x+3) - \log_4(x-1) = 2$

Sol. $\log_4 8 + \log_4(x+3) - \log_4(x-1) = 2$

$$\Rightarrow \quad \log_4 \frac{8(x+3)}{x-1} = 2$$

$$\Rightarrow \quad \frac{8(x+3)}{x-1} = 2^4$$

$$\Rightarrow \quad x + 3 = 2x - 2$$

$$\Rightarrow \quad x = 5$$

Also for $x = 5$ all terms of the equation are defined.

Example 5 Solve: $\log(-x) = 2\log(x+1)$

Sol. By definition, $x < 0$ and $x + 1 > 0 \Rightarrow -1 < x < 0$

Now. $\log(-x) = 2\log(x+1)$

$$\Rightarrow \quad -x = (x+1)^2$$

$$\Rightarrow \quad x^2 + 3x + 1 = 0$$

$$\Rightarrow \quad x = \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$$

Hence, $x = \dfrac{-3+\sqrt{5}}{2}$ is the only solution.

Example 6 Solve: $\log_{x^2+6x+8}(\log_{2x^2+2x+3}(x^2-2x)) = 0$

Sol. $\log_{x^2+6x+8}(\log_{2x^2+2x+3}(x^2-2x)) = 0$

$$\therefore \quad \log_{2x^2+2x+3}(x^2-2x) = 1$$

$$\therefore \quad x^2 - 2x = 2x^2 + 2x + 3$$

$$\Rightarrow \quad x^2 + 4x + 3 = 0$$

$$\Rightarrow \quad (x+1)(x+3) = 0$$

$$\therefore \quad x = -1, -3.$$

But for $x = -3$, $x^2 + 6x + 8 < 0$

$$\therefore \quad x = -1$$

Example 7 Solve: $\log_2\left(2\sqrt{17-2x}\right) = 1 - \log_{1/2}(x-1)$

Sol. $\log_2\left(2\sqrt{17-2x}\right) = 1 + \log_2(x-1)$

$$\Rightarrow \quad \log_2\left(\frac{2\sqrt{17-2x}}{x-1}\right) = 1$$

$\Rightarrow \quad \left(\dfrac{2\sqrt{17-2x}}{x-1}\right) = 2$

$\Rightarrow \quad 2\sqrt{17-2x} = 2(x-1)$

$\Rightarrow \quad x^2 - 2x + 1 = 17 - 2x$

$\Rightarrow \quad x^2 = 16$

$\Rightarrow \quad x = 4$ or -4 (rejected)

Example 8 Solve: $2^{x+2}27^{x/(x-1)} = 9$

Sol. Taking log of both sides, we have $(x+2)\log 2 + \dfrac{x}{x-1}\log 27 = \log 9$

$\Rightarrow \quad (x+2)\log 2 + \dfrac{x}{x-1}3\log 3 = 2\log 3$

$\Rightarrow \quad (x+2)\log 2 + \left(\dfrac{3x}{x-1} - 2\right)\log 3 = 0$

$\Rightarrow \quad (x+2)\left[\log 2 + \dfrac{\log 3}{x-1}\right] = 0$

$\Rightarrow \quad x = -2$ or $x - 1 = -\dfrac{\log 3}{\log 2}$

$\Rightarrow \quad x = -2,\ 1 - \dfrac{\log 3}{\log 2}$

Example 9 Solve: $\log_5(5^{1/x} + 125) = \log_5 6 + 1 + \dfrac{1}{2x}$

Sol. $\log_5(5^{1/x} + 125) = \log_5 6 + 1 + \dfrac{1}{2x}$

$\Rightarrow \quad \log_5(5^{1/x} + 125) - \log_5 6 = 1 + \dfrac{1}{2x}$

$\Rightarrow \quad \log_5\left(\dfrac{5^{\frac{1}{x}} + 125}{6}\right) = 1 + \dfrac{1}{2x}$

$\Rightarrow \quad \dfrac{5^{\frac{1}{x}} + 125}{6} = 5^{1+\frac{1}{2x}}$

$\Rightarrow \quad \dfrac{5^{\frac{1}{x}} + 125}{6} = 5 \times 5^{\frac{1}{2x}}$

$\Rightarrow \quad \dfrac{y^2 + 125}{6} = 5y$ (where $y = 5^{\frac{1}{2x}}$)

$\Rightarrow \quad y^2 - 30y + 125 = 0$

$\Rightarrow \quad y = 5 \text{ or } 25$

$\Rightarrow \quad 5^{\frac{1}{2x}} = 5 \text{ or } 25$

$\Rightarrow \quad x = 1/2 \text{ or } 1/4$

Example 10 Solve: $\log_2(4 \times 3^x - 6) - \log_2(9^x - 6) = 1$

Sol. $\log_2(4 \times 3^x - 6) - \log_2(9^x - 6) = 1$

$$\Rightarrow \quad \log_2 \frac{4 \times 3^x - 6}{9^x - 6} = 1$$

$$\Rightarrow \quad \frac{4 \times 3^x - 6}{9^x - 6} = 2$$

$\Rightarrow \quad 4y - 6 = 2y^2 - 12$ (putting $3^x = y$)

$\Rightarrow \quad y^2 - 2y - 3 = 0$

$\Rightarrow \quad y = -1, 3$

$\Rightarrow \quad 3^x = 3$

$\Rightarrow \quad x = 1$

Example 11 Solve: $6(\log_x 2 - \log_4 x) + 7 = 0$

Sol. $6(\log_x 2 - \log_4 x) + 7 = 0$

$$\Rightarrow \quad 6\left(\log_x 2 - \frac{1}{2}\log_2 x\right) + 7 = 0$$

$$\Rightarrow \quad 6\left(\frac{1}{y} - \frac{y}{2}\right) + 7 = 0$$

$$\Rightarrow \quad 6\left(\frac{2 - y^2}{2y}\right) + 7 = 0$$

$$\Rightarrow \quad 3\left(\frac{2 - y^2}{y}\right) + 7 = 0$$

$\Rightarrow \quad 6 - 3y^2 + 7y = 0$

$\Rightarrow \quad 3y^2 - 7y - 6 = 0$

$\Rightarrow \quad 3y^2 + 2y - 9y - 6 = 0$

$\Rightarrow \quad (y - 3)(3y + 2) = 0$

$\Rightarrow \quad y = 3 \text{ or } y = -2/3$

$\Rightarrow \quad \log_2 x = 3 \text{ or } -2/3$

$\Rightarrow \quad x = 8 \text{ or } x = 2^{-2/3}$

Example 12 Solve: $(\log_3 x)(\log_5 9) - \log_x 25 + \log_3 2 = \log_3 54$

Sol. $(\log_3 x)(\log_5 9) - \log_x 25 + \log_3 2 = \log_3 54$

$$\Rightarrow \quad \frac{\log x}{\log 3} \times \frac{\log 9}{\log 5} - 5\log_x 5 + \log_3 2 = 3 + \log_3 2$$

$\Rightarrow \quad 2\log_5 x - 2\log_x 5 + \log_3 2 = 3 + \log_3 2$

$\Rightarrow \quad 2t^2 - 2 = 3t$ (putting $\log_5 x = t$)

$\Rightarrow \quad 2t^2 - 3t - 2 = 0$

$\Rightarrow \quad (2t + 1)(t - 2) = 0$

$\Rightarrow \quad t = -1/2$ or $t = 2$

$\Rightarrow \quad \log_5 x = -1/2$ or $\log_5 x = 2$

$\Rightarrow \quad x = 1/\sqrt{5} \quad x = 25$

Example 13 Solve: $4^{\log_2 \log x} = \log x - (\log x)^2 + 1$

Sol. $\log_2 \log x$ is meaningful if $x > 1$

Since $4^{\log_2 \log x} = 2^{2\log_2 \log x} = (2^{\log_2 \log x})^2$

$= (\log x)^2 \qquad (a^{\log_a x} = x, a > 0, a \neq 1)$

So the given equation reduces to

$2(\log x)^2 - \log x - 1 = 0$

$\Rightarrow \quad \log x = 1$, or $\log x = -1/2$.

But for $x > 1$, $\log x > 0$, so $\log x = 1$ i.e. $x = e$.

Example 14 Solve: $\sqrt{\log_2 x} - 0.5 = \log_2 \sqrt{x}$

Sol. $\sqrt{\log_2 x} - 0.5 = \log_2 \sqrt{x}$

$\Rightarrow \quad \sqrt{\log_2 x} - 0.5 = 0.5\log_2 x$

$\Rightarrow \quad y - 0.5 = 0.5y^2$

$\Rightarrow \quad y^2 - 2y + 1 = 0 \Rightarrow y = 1 \Rightarrow \log_2 x = 1 \Rightarrow x = 2$

Example 15 Solve: $4\log_{x/2}\left(\sqrt{x}\right) + 2\log_{4x}\left(x^2\right) = 3\log_{2x}\left(x^3\right)$

Sol. $$\frac{4\log_2 \sqrt{x}}{\log_2(x/2)} + \frac{2\log_2(x^2)}{\log_2(4x)} = \frac{3\log_2(x^3)}{\log_2(2x)}$$

$$\Rightarrow \quad \frac{4\cdot\frac{1}{2}\log_2(x)}{\log_2 x - 1} + \frac{4\log_2(x)}{2+\log_2(x)} = \frac{9\log_2(x)}{1+\log_2(x)}$$

Let $\log_2 x = t$

Given equation reduces to $\dfrac{2t}{t-1} + \dfrac{4t}{t+2} = \dfrac{9t}{t+1}$

$$\Rightarrow \quad t = 0 \text{ or } \frac{2}{t-1} + \frac{4}{t+2} = \frac{9}{t+1}$$

$$\Rightarrow \quad \frac{2t+4+4t-4}{(t-1)(t+2)} = \frac{9}{t+1}$$

$\Rightarrow \quad t^2 + t - 6 = 0$

$\Rightarrow \quad (t+3)(t-2) = 0$

$\Rightarrow \quad t = 0, t = 2 \text{ or } t = -3$

$\Rightarrow \quad x = 1, x = 4, x = 1/8$

Example 16 Solve: $\log_{3\sqrt{x}} x + \log_{3x} \sqrt{x} = 0$

Sol. $\dfrac{\log x}{\log 3 + (1/2)\log x} + \dfrac{(1/2)\log x}{\log 3 + \log x}$

$$\Rightarrow \quad \frac{\log_3 x}{1 + (1/2)\log_3 x} + \frac{1}{2}\frac{\log_3 x}{(1+\log_3 x)} = 0$$

Let $\log_3 x = y$

$$\Rightarrow \quad \frac{y}{1+(y/2)} + \frac{y}{2(1+y)} = 0$$

$$\Rightarrow \quad y\left(\frac{2}{2+y} + \frac{1}{2(1+y)}\right) = 0$$

$\Rightarrow \quad y[4 + 4y + 2 + y] = 0$

$\Rightarrow \quad y = 0 \text{ or } y = -6/5$

$\Rightarrow \quad \log_3 x = 0 \text{ or } \log_3 x = -6/5$

$\Rightarrow \quad x = 1 \text{ or } x = 3^{-6/5}$

Example 17 Solve: $4^{\log_9 x} - 6 \cdot x^{\log_9 2} + 2^{\log_3 27} = 0.$

Sol. Let $2^{\log_9 x} = y$

we get $y^2 - 6y + 8 = 0$

$\Rightarrow \quad y = 4 \text{ or } 2$

If $2^{\log_9 x} = 2^2 \Rightarrow \log_9 x = 2$

$\Rightarrow \quad x = 81$

If $2^{\log_9 x} = 2^1 \Rightarrow \log_9 x = 1 \Rightarrow x = 9$

Example 18 Solve: $(x^{\log_{10} 3})^2 - (3^{\log_{10} x}) - 2 = 0$

Sol. Let $(x^{\log_{10} 3}) = (3^{\log_{10} x}) = t$

Thus, given equation becomes

$t^2 - t - 2 = 0$

$\Rightarrow \quad (t-2)(t+1) = 0$

$\Rightarrow \quad t = 2;$ (as $t = -1$ is not possible)

$\Rightarrow \quad (3^{\log_{10} x}) = 2$

$\Rightarrow \quad \log_{10} x = \log_3 2$

$\Rightarrow \quad x = 10^{\log_3 2}$

Example 19 Solve: $3^{(\log_9 x)^2 - \frac{9}{2}\log_9 x + 5} = 3\sqrt{3}$

Sol. We have

$$(\log_9 x)^2 - \frac{9}{2}\log_9 x + 5 = \frac{3}{2}$$

Put $\log_9 x = y$

$\Rightarrow \quad y^2 - \frac{9}{2}y + 5 = \frac{3}{2}$

$\Rightarrow \quad 2y^2 - 9y + 10 = 3$

$\Rightarrow \quad 2y^2 - 9y + 7 = 0$

$\Rightarrow \quad (2y - 7)(y - 1) = 0$

$\Rightarrow \quad y = \frac{7}{2}, 1$

$\therefore \quad$ Either $\log_9 x = 1$ or $\log_9 x = \frac{7}{2}$

$\Rightarrow \quad$ Either $x = 9$ or $x = 9^{7/2} = 3^7$

Example 20 Solve for x: $(2x)^{\log_b 2} = (3x)^{\log_b 3}$

Sol. $(2x)^{\log_b 2} = (3x)^{\log_b 3}$

$\Rightarrow \quad \log_b 2[\log 2 + \log x] = \log_b 3[\log 3 + \log x]$

$\Rightarrow \quad (\log_b 2)(\log 2) - \log_b 3 \cdot \log 3 = (\log_b 3 - \log_b 2)\log x$

$\Rightarrow \quad \frac{\log 2}{\log b} \cdot \log 2 - \frac{\log 3}{\log b} \cdot \log 3 = \left(\frac{\log 3}{\log b} - \frac{\log 2}{\log b}\right)\log x$

$\Rightarrow \quad \frac{(\log_2)^2 - (\log_3)^3}{\log b} = \left(\frac{\log 3 - \log 2}{\log b}\right)\log x$

$\Rightarrow \quad \log x = -(\log 3 + \log 2) = \log(6)^{-1}$

$\Rightarrow \quad x = \frac{1}{6}$

Example 21 If the equation $2^x + 4^y = 2^y + 4^x$ is solved for y in terms of x where $x < 0$, then find the sum of the solutions.

Sol. $2^{2y} - 2^y + 2^x(1 - 2^x) = 0$

Putting $2^y = t$, we get

$t^2 - t + 2^x(1 - 2^x) = 0$

$\therefore \quad t_1 t_2 = 2^x(1 - 2^x) \qquad$ where $t_1 = 2^{y_1}$ and $t_2 = 2^{y_2}$

$\Rightarrow \quad 2^{y_1 + y_2} = 2^x(1 - 2^x)$

$\Rightarrow \quad y_1 + y_2 = x + \log_2(1 - 2^x)$

Example 22 Solve: $\log_{10}\left[\frac{1}{2^x+x-1}\right] = x\,[\log_{10}5 - 1]$

Sol. R.H.S $= x\,[\log_{10}5 - \log_{10}10] = x\log_{10}\frac{5}{10} = \log_{10}\frac{1}{2^x}$

$\therefore \quad \frac{1}{2^x+x-1} = \frac{1}{2^x}$

$\therefore \quad x - 1 = 0$ or $x = 1$

Example 23 Solve: $a^{2\log_2 x} = 5 + 4x^{\log_2 a}$, where $a > 1$.

Sol. Given equation can be written as $(a^{\log_2 x})^2 = 5 + 4\,a^{\log_2 x}$

Let $a^{\log_2 x} = t$

Thus, given equation reduces to $t^2 - 4t - 5 = 0$.

$\Rightarrow \quad (t-5)(t+1) = 0$

$\Rightarrow \quad t = 5$ or $t = -1$ (rejected)

$\therefore \quad a^{\log_2 x} = 5$

$\Rightarrow \quad x^{\log_2 a} = 5$

$\Rightarrow \quad x = 5^{\log_a 2}$

Example 24 Solve: $\log_{x+1}(x-0.5) = \log_{x-0.5}(x+1)$

Sol. $\frac{\log_2(x-0.5)}{\log_2(x+1)} = \frac{\log_2(x+1)}{\log_2(x-0.5)}$

$\Rightarrow \quad [\log_2(x+1)]^2 = [\log_2(x-0.5)^2]$

$\Rightarrow \quad \log_2(x+1) = \log_2(x-0.5)$ or $-\log_2(x-0.5)$

If $\log_2(x+1) = \log_2(x-0.5) \quad \Rightarrow \quad x+1 = x-0.5 \quad \Rightarrow$ no solution

If $\log_2(x+1) = \log(x-0.5)^{-1}$

$\Rightarrow \quad x + 1 = \frac{1}{x-(1/2)} = \frac{2}{2x-1}$

$\Rightarrow \quad (x+1)(2x-1) = 2$

$\Rightarrow \quad 2x^2 + x - 3 = 0$

$\Rightarrow \quad 2x^2 + 3x - 2x - 3 = 0$

$\Rightarrow \quad (x-1)(2x+3) = 0$

$\Rightarrow \quad x = 1 \; (\because x = -3/2$ rejected)

Example 25 Solve the system of equations:

$(\log_a x)(\log_a(xyz)) = 48$

$(\log_a y)(\log_a(xyz)) = 12$

$(\log_a z)(\log_a(xyz)) = 84$

Sol. On adding given equations, we get

$\Rightarrow \quad \log_a(xyz)\,[\log_a x + \log_a y + \log_a z] = 144$

$\Rightarrow \quad \log_a(xyz) = (144)^{1/2} = 12$

$\Rightarrow \quad xyz = a^{12}$

From $\log_a x \log_a(xyz) = 48$

$\Rightarrow \quad (\log_a x)(12) = 48$

$\Rightarrow \quad \log_a x = 4$

$\Rightarrow \quad x = a^4$

Similarly, $y = a$ and $z = a^7$.

Example 26 Solve: $\left(\sqrt{\pi}\right)^{\log_\pi x} \cdot \left(\sqrt{\pi}\right)^{\log_{\pi^2} x} \cdot \left(\sqrt{\pi}\right)^{\log_{\pi^4} x} \cdot \left(\sqrt{\pi}\right)^{\log_{\pi^8} x} \ldots\ldots \infty = 3$

Sol. We have, $\left(\sqrt{\pi}\right)^{\log_\pi x} \cdot \left(\sqrt{\pi}\right)^{\log_{\pi^2} x} \cdot \left(\sqrt{\pi}\right)^{\log_{\pi^4} x} \cdot \left(\sqrt{\pi}\right)^{\log_{\pi^8} x} \ldots\ldots \infty = 3$

$$\Rightarrow \quad \left(\sqrt{\pi}\right)^{\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots\infty\right)\log_\pi x} = 3$$

$$\Rightarrow \quad \left(\sqrt{\pi}\right)^{2\log_\pi x} = 3$$

$$\Rightarrow \quad \pi^{\log_\pi x} = 3$$

$\Rightarrow \quad x = 3$

Example 27 Solve: $2^{\log_a(2x)} = 5^{\log_a(5x)}$; If $a > 1$

Sol. We have $2^{\log_a(2x)} = 5^{\log_a(5x)}$

Taking log on both sides, we get

$\log_a(2x) \,.\, \log 2 = \log_a(5x) \,.\, \log 5$

$$\Rightarrow \quad \frac{(\log 2 + \log x)}{\log a} \log 2 = \frac{(\log 5 + \log x)}{\log a} \log 5$$

$\Rightarrow \quad (\log 2)^2 + \log x .\ \log 2 = (\log 5)^2 + (\log x) \log 5$

$\Rightarrow \quad \log x\,(\log 2 - \log 5) = (\log 5)^2 - (\log 2)^2$

$\Rightarrow \quad -\log x = \log 5 + \log 2 = \log 10$

$$\Rightarrow \quad x = \frac{1}{10}$$

Example 28 Prove that the equation $x^{\log_{\sqrt{x}} 2x} = 4$ has no solution.

Sol. We have $x^{\log_{\sqrt{x}} 2x} = 4$, $x > 0, x \neq 1$

$\Rightarrow \quad x^{2\log_x 2x} = 4$

$\Rightarrow \quad x^{\log_x 4x^2} = 4$

$\Rightarrow \quad 4x^2 = 4$

$\Rightarrow \quad x = \pm 1$, which is not possible.

Hence equation has no solution.

Example 29 Solve: $x^{\log_x(1-x)^2} = 9$

Sol. Taking logarithm of both the sides with base 3, we have

$$\log_x (1-x)^2 \log_3 x = 2$$

$$\Rightarrow \quad \frac{\log_3(1-x)^2}{\log_3 x} \log_3 x = 2$$

$$\Rightarrow \quad \log_3 (1-x)^2 = 2$$

$$\Rightarrow \quad (1-x)^2 = 9 \text{ (clearly } x \neq 3)$$

$$\Rightarrow \quad x = 4, -2$$

But $x > 0$, So the solution set is $\{4\}$.

Example 30 Solve: $(x+1)^{\log_{10}(x+1)} = 100(x+1)$

Sol. $(x+1)^{\log_{10}(x+1)} = 100(x+1)$

$$\Rightarrow \quad \log_{10}(x+1)\log_{10}(x+1) = \log_{10}(100(x+1))$$

$$\Rightarrow \quad \log_{10}(x+1)\log_{10}(x+1) = 2 + \log_{10}(x+1)$$

Let $\log_{10}(x+1) = y$. Then

$$y^2 - y - 2 = 0$$

$$\Rightarrow \quad y = 2 \text{ or } -1$$

$$\Rightarrow \quad \log_{10}(x+1) = 2 \text{ or } -1$$

$$\Rightarrow \quad x + 1 = 100 \text{ or } 1/10 \Rightarrow x = 99 \text{ or } -9/10$$

Example 31 Solve: $\sqrt[4]{|x-3|^{x+1}} = \sqrt[3]{|x-3|^{x-2}}$

Sol. $\sqrt[4]{|x-3|^{x+1}} = \sqrt[3]{|x-3|^{x-2}}$

Taking log on both the sides, we get

$$\frac{x+1}{4}\log|x-3| = \frac{x-2}{3}\log|x-3|$$

$$\Rightarrow \quad \log|x-3|\left[\frac{x+1}{4} - \frac{x-2}{3}\right] = 0$$

$$\Rightarrow \quad \log|x-3| = 0 \text{ or } \left[\left(\frac{x+1}{4}\right) - \left(\frac{x-2}{3}\right)\right] = 0$$

$$\Rightarrow \quad x = 4, 2 \text{ or } x = 11$$

Example 32 Solve: $5 \cdot 3^{\log_3 x} - 2^{1-\log_2 x} - 3 = 0$

Sol. $5 \cdot 3^{\log_3 x} - 2^{1-\log_2 x} - 3 = 0$

$$\Rightarrow \quad 5x - \frac{2}{x} - 3 = 0$$

$\Rightarrow \quad 5x^2 - 3x - 2 = 0$

$\Rightarrow \quad x = 1 \text{ or } x = \dfrac{-2}{5}$ (not possible)

$\Rightarrow \quad x = 1$

Example 33 Solve: $|x-1|^{(\log_{10}x)^2 - \log_{10}x^2} = |x-1|^3$

Sol. We have $|x-1|^{(\log_{10}x)^2 - \log_{10}x^2} = |x-1|^3, x > 0, x \neq 1$

$\Rightarrow \quad |x-1| = 1 \text{ or } (\log_{10}x)^2 - \log_{10}x^2 = 3$

$\Rightarrow \quad x = 2 \text{ or } (\log_{10}x)^2 - 2\log_{10}x - 3 = 0$

$\Rightarrow \quad x = 2 \text{ or } (\log_{10}x - 3)(\log_{10}x + 1) = 0$

$\Rightarrow \quad x = 2 \text{ or } \log_{10}x = 3, \log_{10}x = -1$

$\Rightarrow \quad x = 2, x = 1000 \text{ or } x = 0.1$

Example 34 Solve: $x^{\log_3 x^2 + (\log_3 x)^2 - 10} = 1/x^2$

Sol. Taking log of both the sides with base 3, we have

$(\log_3 x^2 + (\log_3 x)^2 - 10)(\log_3 x) = -2\log_3 x$

$\Rightarrow \quad \log_3 x = 0 \text{ or } 2\log_3 x + (\log_3 x)^2 - 8 = 0$

$\Rightarrow \quad x = 1, \log_3 x = -1 \pm 3 \text{ or } \log_3 x = 2, \log_3 x = -4.$

Hence $x = 1, 3^2, 3^{-4} = 1, 9, 1/81$.

Example 35 Solve: $x^{\sqrt{x}} = \left(\sqrt{x}\right)^x$

Sol. $x^{\sqrt{x}} = \left(\sqrt{x}\right)^x$

$\Rightarrow \quad \sqrt{x}\,\log x = x\log\sqrt{x}$

$\Rightarrow \quad \log x\left[\sqrt{x} - \dfrac{x}{2}\right] = 0$

$\Rightarrow \quad \log x = 0 \text{ or } \left[\sqrt{x} - \dfrac{x}{2}\right] = 0$

$\Rightarrow \quad x = 1 \text{ or } x = 4$

Exercise 1

1. Solve: $\log_2(x-3) = 3$
2. Solve: $\log_7(x-2) + \log_7(x+3) = \log_7 14$
3. Solve: $\log_2(x+5) - \log_2(2x-1) = 5$
4. Solve: $\log_2(3x-2) = \log_{1/2} x$
5. Solve: $2\log_{2+\sqrt{3}}\left(\sqrt{x^2+1} + x\right) + \log_{2-\sqrt{3}}\left(\sqrt{x^2+1} - x\right) = 3$

6. Solve: $\log_4 (x-1) = \log_2 (x-3)$
7. Solve: $\log_6 9 - \log_9 27 + \log_8 x = \log_{64} x - \log_6 4$
8. Solve: $\sqrt{\log(-x)} = \log\sqrt{x^2}$ (base is 10)
9. If $\log_y x + \log_x y = 2$, $x^2 + y = 12$, then find the value of xy.
10. Find the value of b for which the equation $2\log_{1/25}(bx+28) = -\log_5(12-4x-x^2)$ has coincident roots.
11. Solve: $\log_3 \{5 + 4\log_3 (x-1)\} = 2$
12. Solve: $\log_4 (3-x) + \log_{0.25} (3+x) = \log_4 (1-x) + \log_{0.25} (2x+1)$
13. Solve: $\dfrac{\log_8 (8/x^2)}{(\log_8 x)^2} = 3$
14. Find value/values of the parameter k for which $(\log_{16} x)^2 - \log_{16} x + \log_{16} k = 0$ with real coefficients will have exactly one solution.
15. Solve: If $\log_3 (5x-2) - 2\log_3 \sqrt{3x+1} = 1 - \log_3 4$
16. Solve: $\dfrac{3}{2}\log_4(x+2)^2 + 3 = \log_4(4-x)^3 + \log_4(6+x)^3$
17. Solve: $\begin{vmatrix} \log_{10} a & -1 \\ \log_{10}(a-1) & 2 \end{vmatrix} = \log_{10} a + \log_{10} 2$
18. Solve: $\log_5\left(\log_{64}|x| + (25)^x - \dfrac{1}{2}\right) = 2x$
19. Solve: $\log_2(x^2 - x)\log_2\left(\dfrac{x-1}{x}\right) + (\log_2 x)^2 = 4$
20. Solve: $\log_2(25^{x+3} - 1) = 2 + \log_2(5^{x+3} + 1)$
21. Solve: $\log_4(2 \times 4^{x-2} - 1) + 4 = 2x$

Exercise 2

1. Solve: $\log_x 2 \log_{2x} 2 = \log_{4x} 2$
2. Solve: $\log_{2x} 2 + \log_4 2x = -\dfrac{3}{2}$
3. Solve: $\log_{ax} a + \log_x a^2 + \log_{a^2x} a^3 = 0$; $a > 0, \neq 1$
4. If $\log_2 x + \log_x 2 = \dfrac{10}{3} = \log_2 y + \log_y 2$ and $x \neq y$ then find the value of $x + y$.
5. If $xy^2 = 4$ and $\log_3 (\log_2 x) + \log_{1/3} (\log_{1/2} y) = 1$, then find the value of x.
6. Solve: $2^{x+2} . 5^{6-x} = 10^{x^2}$
7. Solve: $\sqrt{1 + \log_x \sqrt{27}}\ \log_3 x + 1 = 0$

8. If the equation $x^{\log_a x^2} = \dfrac{x^{k-2}}{a^k}$, $a \neq 0$, has exactly one solution for x, then find the values of k.
9. Solve: $7^{\log_7(x^2-4x+5)} = (x-1)$
10. Solve: $x^{\log_{10} x} = 100x$
11. Solve: $\log_3 x \cdot \log_4 x \cdot \log_5 x = \log_3 x \cdot \log_4 x + \log_4 x \cdot \log_5 x + \log_5 x \cdot \log_3 x$.
12. Solve: $\log_{3/4} \log_8 (x^2+7) + \log_{1/2} \log_{1/4} (x^2+7)^{-1} = -2$
13. If $5^{\log x} - 3^{\log x - 1} = 3^{\log x + 1} - 5^{\log x - 1}$, where the base of logarithm is 10, then find the value of x.
14. Solve: $\log_{x^2} 16 + \log_{2x} 64 = 3$
15. Solve: $9^{\log_3(\log_e x)} = \log_e x - (\log_e x)^2 + 1$
16. Solve the equations for x and y: $(3x)^{\log 3} = (4y)^{\log 4}$, $4^{\log x} = 3^{\log y}$
17. Solve: $x^{\log_y x} = 2$ and $y^{\log_x y} = 16$

CHAPTER

5

Logarithmic Inequalities

Generalized Method of Intervals for Solving Inequalities

Let $F(x) = (x-a_1)^{k_1}(x-a_2)^{k_2}\dots(x-a_{n-1})^{k_{n-1}}(x-a_n)^{k_n}$ where $k_1, k_2, \dots k_n \in Z$ and $a_1, a_2, \dots a_n$ are fixed real numbers satisfying the condition

$$a_1 < a_2 < a_3 < \dots < a_{n-1} < a_n$$

For solving $F(x) > 0$ or $F(x) < 0$, consider the following algorithm:

- We mark the numbers $a_1, a_2, \dots, a_n$ on the number axis and put the plus sign in the interval on the right of the largest of these numbers, i.e., on the right of a_n.
- Then we put the plus sign in the interval on the left of a_n if k_n is an even number and the minus sign if k_n is an odd number. In the next interval, we put a sign according to the following rule:
- When passing through the point a_{n-1} the polynomial $F(x)$ changes sign if k_{n-1} is an odd number. Then we consider the next interval and put a sign in it using the same rule.
- Thus we consider all the intervals. The solution of the inequality $F(x) > 0$ is the union of all intervals in which we have put the plus sign and the solution of the inequality $F(x) < 0$ is the union of all intervals in which we have put the minus sign.

Frequently used Inequalities

(i) $(x-a)(x-b) < 0 \Rightarrow x \in (a, b)$, where $a < b$

(ii) $(x-a)(x-b) > 0 \Rightarrow x \in (-\infty, a) \cup (b, \infty)$, where $a < b$

(iii) $x^2 \le a^2 \Rightarrow x \in [-a, a]$

(iv) $x^2 \ge a^2 \Rightarrow x \in (-\infty, -a] \cup [a, \infty)$

(v) If $ax^2 + bx + c < 0$, $(a > 0) \Rightarrow x \in (\alpha, \beta)$, where $\alpha, \beta\ (\alpha < \beta)$ are the roots of the equation $ax^2 + bx + c = 0$

(vi) If $ax^2 + bx + c > 0$, $(a > 0) \Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$, where $\alpha, \beta\ (\alpha < \beta)$ are roots of the equation $ax^2 + bx + c = 0$

Example 1 Solve: $(2x + 1)(x - 3)(x + 7) < 0$

Sol. $(2x + 1)(x - 3)(x + 7) < 0$

Sign scheme of $(2x + 1)(x - 3)(x + 7)$ is as follows:

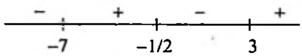

Fig. 5.1

Hence, solution is $(-\infty, -7) \cup (-1/2, 3)$.

Example 2 Solve: $\dfrac{2}{x} < 3$

Sol. $\dfrac{2}{x} < 3$

$$\Rightarrow \quad \frac{2}{x} - 3 < 0$$

(we cannot cross multiply with x as x can be negative or positive)

$$\Rightarrow \quad \frac{2 - 3x}{x} < 0$$

$$\Rightarrow \quad \frac{3x - 2}{x} > 0$$

$$\Rightarrow \quad \frac{(x - 2/3)}{x} > 0$$

Sign scheme of $\dfrac{(x - 2/3)}{x}$ is as follows:

Fig. 5.2

$$\Rightarrow \quad x \in (-\infty, 0) \cup (2/3, \infty)$$

Example 3 Solve: $\dfrac{2x - 3}{3x - 5} \geq 3$

Sol. $\dfrac{2x - 3}{3x - 5} \geq 3$

$$\Rightarrow \quad \frac{2x - 3}{3x - 5} - 3 \geq 0$$

$$\Rightarrow \quad \frac{2x - 3 - 9x + 15}{3x - 5} \geq 0$$

$\Rightarrow \qquad \dfrac{-7x+12}{3x-5} \geq 0$

$\Rightarrow \qquad \dfrac{7x-12}{3x-5} \leq 0$

Sign scheme of $\dfrac{7x-12}{3x-5}$ is as follows:

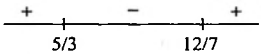

Fig. 5.3

$\Rightarrow \qquad x \in (5/3, 12/7]$

$x = 5/3$ is not included in the solution as at $x = 5/3$, denominator becomes zero.

Example 4 Solve: $x^2 - x - 1 < 0$

Sol. Let's first factorize $x^2 - x - 1$

For that let $x^2 - x - 1 = 0$

$$\Rightarrow \qquad x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Now on number line(x-axis) mark $x = \dfrac{1 \pm \sqrt{5}}{2}$

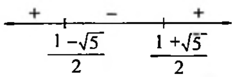

Fig. 5.4

From the sign scheme of $x^2 - x - 1$ which is shown in above figure,

$x^2 - x - 1 < 0$

$$\Rightarrow \qquad x \in \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$$

Example 5 Solve: $(x-1)^2 (x+4) < 0$

Sol. $(x-1)^2 (x+4) < 0$ (1)

Sign scheme of $(x-1)^2 (x+4)$ is as follows

Fig. 5.5

Sign of expression does not change at $x = 1$ as factor $(x - 1)$ has even power.

Hence solution of (1) is $x \in (-\infty, -4)$.

Example 6 Solve: $\dfrac{x(3-4x)(x+1)}{(2x-5)}<0$

Sol. $\dfrac{x(3-4x)(x+1)}{(2x-5)}<0$

$$\Rightarrow \quad -\frac{x(4x-3)(x+1)}{(2x-5)}<0$$

$$\Rightarrow \quad \frac{x(4x-3)(x+1)}{(2x-5)}>0$$

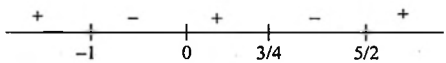

Fig. 5.6

$$\Rightarrow \quad x \in (-\infty, -1) \cup (0, 3/4) \cup (5/2, \infty)$$

Example 7 Solve: $\dfrac{2x}{2x^2+5x+2} > \dfrac{1}{x+1}$

Sol. We are given $\dfrac{2x}{2x^2+5x+2} > \dfrac{1}{x+1}$

$$\Rightarrow \quad \frac{2x}{2x^2+5x+2} - \frac{1}{x+1} > 0$$

$$\Rightarrow \quad \frac{2x^2+2x-2x^2-5x-2}{(2x^2+5x+2)(x+1)} > 0$$

$$\Rightarrow \quad \frac{-3x-2}{(2x+1)(x+1)(x+2)} > 0$$

$$\Rightarrow \quad \frac{(3x+2)}{(x+1)(x+2)(2x+1)} < 0$$

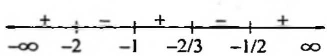

Fig. 5.7

From the sign scheme solution is: $x \in (-2, -1) \cup (-2/3, -1/2)$

Absolute Value of *x*

The absolute value of a real number 'x' is that number's distance from zero along the real number line and it is denoted by $|x|$.

The absolute value of x, denoted by "$|x|$" (and which is read as "the absolute value of x"), is the distance of x from zero. This is why absolute value is never negative; absolute value only asks "how far?", not "in which direction?". This means not only that $|3| = 3$, because 3 is three units to the right of zero, but also that $|-3| = 3$, because -3 is three units to the left of zero.

Thus $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Also, $\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$|x-a| = \begin{cases} x-a, & x \geq a \\ a-x, & x<a \end{cases}$, where $a > 0$

In general, $|f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x)<0 \end{cases}$,

where $y = f(x)$ is any real valued function.

Graph of function $f(x) = y = |x|$

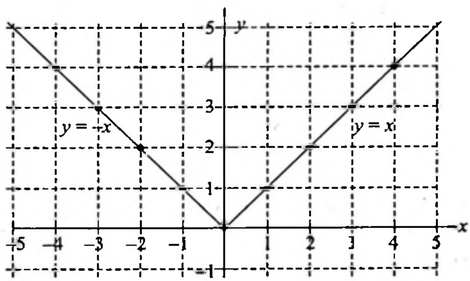

Fig. 5.8

We can see that graph of $y = |x|$ is in 1st and 2nd quadrants only where $y \geq 0$, hence $|x| \geq 0$.

Example 8 Solve the following:

(i) $|x| = 3$ (ii) $x^2 - |x| - 2 = 0$

Sol. (i) $|x| = 3$, i.e., those points on real number line which are at distance 3 units from 0, which are -3 and 3.

Thus $|x| = 3 \Rightarrow x = \pm 3$

(ii) $x^2 - |x| - 2 = 0$

$\Rightarrow \quad |x|^2 - |x| - 2 = 0$

$\Rightarrow \quad (|x| - 2)(|x| + 1) = 0$

$\Rightarrow \quad |x| = 2 \; (\because |x| + 1 \neq 0)$

$\Rightarrow \quad x = \pm 2$

Example 9 Solve the following:

(i) $|x - 2| = 1$ (ii) $2|x+1|^2 - |x+1| = 3$

Sol. (i) $|x - 2| = 1$, i.e., those points on real number line which are at distance 1 unit from 2.

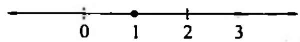

Fig. 5.9

As shown in the figure $x = 1$ and $x = 3$ are at distance 1 unit from 2,

Hence $x = 1$ or $x = 3$.

Thus $|x - 2| = 1$

$\Rightarrow \quad x - 2 = \pm 1$

$\Rightarrow \quad x = 1$ or $x = 3$

(ii) $2|x+1|^2 - |x+1| = 3$

$\Rightarrow \quad 2|x+1|^2 - |x+1| - 3 = 0$

$\Rightarrow \quad 2|x+1|^2 - 3|x+1| + 2|x+1| - 3 = 0$

$\Rightarrow \quad (2|x+1| - 3)(|x+1| + 1) = 0$

$\Rightarrow \quad 2|x+1| - 3 = 0$

$\Rightarrow \quad |x+1| = 3/2$

$\Rightarrow \quad x + 1 = \pm 3/2$

$\Rightarrow \quad x = 1/2$ or $x = -5/2$

Inequalities Involving Absolute Value

(i) $|x| \leq a$ (where $a > 0$)

So, we have to consider those values of x on real number line which are at distance a or less than a from 0.

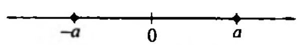

Fig. 5.10

$\Rightarrow \quad -a \leq x \leq a$

e.g., $|x| \leq 2 \Rightarrow -2 \leq x \leq 2$;

$|x| < 3 \Rightarrow -3 < x < 3$ etc.

In general, $|f(x)| \leq a$ (where $a > 0$) $\Rightarrow -a \leq f(x) \leq a$

(ii) $|x| \geq a$ (where $a > 0$)

So, we have to consider those values of x on real number line which are at distance a or more than a from 0.

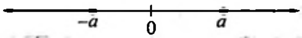

Fig. 5.11

$\Rightarrow \quad x \leq -a$ or $x \geq a$

e.g. $|x| \geq 3 \Rightarrow x \leq -3$ or $x \geq 3$;

$|x| > 2 \Rightarrow x < -2$ or $x > 2$

In general, $|f(x)| \geq a \Rightarrow f(x) \leq -a$ or $f(x) \geq a$

(iii) $a \leq |x| \leq b$ (where $a, b > 0$)

So, we have to consider those values of x on real number line which are at distance equal to a and b or between a and b from 0.

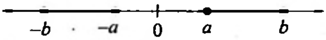

Fig. 5.12

$\Rightarrow \quad x \in [-b, -a] \cup [a, b]$

e.g. $2 \leq |x| \leq 4 \Rightarrow x \in [-4, -2] \cup [2, 4]$

Example 10 $\quad 0 < |x| < 2$

Sol. We know that $|x| \geq 0, \forall x \in R$

But given $|x| > 0 \Rightarrow x \neq 0$

Now, $0 < |x| < 2$

$\Rightarrow \quad x \in (-2, 2), x \neq 0$

$\Rightarrow \quad x \in (-2, 2) - \{0\}$

Example 11 $\quad$ Solve: $|3x - 2| < 4$

Sol. $|3x - 2| < 4$

$\Rightarrow \quad -4 < 3x - 2 < 4$

$\Rightarrow \quad -2 < 3x < 6$

$\Rightarrow \quad -2/3 < x < 2$

Example 12 $\quad$ Solve: $x^2 - 4|x| + 3 < 0$

Sol. $x^2 - 4|x| + 3 < 0$

$\Rightarrow \quad (|x| - 1)(|x| - 3) < 0$

$\Rightarrow \quad 1 < |x| < 3$

$\Rightarrow \quad -3 < x < -1$ or $1 < x < 3$

$\Rightarrow \quad x \in (-3, -1) \cup (1, 3)$

Example 13 $\quad$ Solve: $|x - 3| \geq 2$

Sol. $|x - 3| \geq 2$

$\Rightarrow \quad x - 3 \leq -2$ or $x - 3 \geq 2$

$\Rightarrow \quad x \leq 1$ or $x \geq 5$

Standard Logarithmic Inequalities

(i) If $\log_a x > \log_a y \Rightarrow \begin{cases} x > y, & \text{if } a > 1 \\ x < y, & \text{if } 0 < a < 1 \end{cases}$

(ii) If $\log_a x > y \Rightarrow \begin{cases} x > a^y, & \text{if } a > 1 \\ x < a^y, & \text{if } 0 < a < 1 \end{cases}$

(iii) $\log_a x > 0 \Rightarrow x > 1$ and $a > 1$

or $0 < x < 1$ and $0 < a < 1$

Example 14 Solve: $2^{x+2} - 2^{x+3} - 2^{x+4} > 5^{x+1} - 5^{x+2}$

Sol. $2^{x+2} - 2^{x+3} - 2^{x+4} > 5^{x+1} - 5^{x+2}$

$\Rightarrow \quad 2^x(4 - 8 - 16) > 5^x(5 - 25)$

$\Rightarrow \quad (2/5)^x < 1$

$\Rightarrow \quad x \in (0, \infty)$

Example 15 Solve: $\left(\frac{3}{4}\right)^{6x+10-x^2} < \frac{27}{64}$

Sol. $\left(\frac{3}{4}\right)^{6x+10-x^2} < \frac{27}{64}$

$\Rightarrow \quad \left(\frac{3}{4}\right)^{6x+10-x^2} < \left(\frac{3}{4}\right)^3$

$\Rightarrow \quad 6x + 10 - x^2 > 3$

$\therefore \quad x^2 - 6x - 7 < 0$

$\therefore \quad (x + 1)(x - 7) < 0$

Example 16 Solve: $\log_2(x - 1) > 4$

Sol. $\log_2(x - 1) > 4$

$\Rightarrow \quad x - 1 > 2^4$

$\Rightarrow \quad x > 17$

Example 17 Solve: $\log_3(x - 2) \leq 2$

Sol. $\log_3(x - 2) \leq 2$

$\Rightarrow \quad 0 < x - 2 \leq 3^2$

$\Rightarrow \quad 2 < x \leq 11$

Example 18 Solve: $\log_{0.3}(x^2 - x + 1) > 0$

Sol. $\log_{0.3}(x^2 - x + 1) > 0$

$\Rightarrow \quad 0 < x^2 - x + 1 < (0.3)^0$

$\Rightarrow \quad 0 < x^2 - x + 1 < 1$

$\Rightarrow \quad x^2 - x + 1 > 0$ and $x^2 - x < 0$

$\Rightarrow \quad x(x-1) < 0$

$\Rightarrow \quad 0 < x < 1$ (as $x^2 - x + 1 = (x - 1/2)^2 + 3/4 > 0$ for all real x)

Example 19 Solve: $\log_{10}(x^2 - 2x - 2) \le 0$

Sol. $\log_{10}(x^2 - 2x - 2) \le 0$

$\Rightarrow \quad 0 < x^2 - 2x - 2 \le 10^0$

$\Rightarrow \quad x^2 - 2x - 2 > 0$ and $x^2 - 2x - 3 \le 0$

$\Rightarrow \quad x < 1 - \sqrt{3}$ or $x > 1 + \sqrt{3}$ and $(x-3)(x+1) \le 0$

$\Rightarrow \quad x < 1 - \sqrt{3}$ or $x > 1 + \sqrt{3}$ and $-1 \le x \le 3$

$\Rightarrow \quad x \in [-1, 1-\sqrt{3}) \cup (1 + \sqrt{3}, 3]$

Example 20 Solve: $\log_{0.2}|x - 3| \ge 0$

Sol. $\log_{0.2}|x - 3| \ge 0$

$\Rightarrow \quad 0 < |x - 3| \le (0.2)^0$

$\Rightarrow \quad 0 < |x - 3| \le 1$

$\Rightarrow \quad -1 < x - 3 < 1$ and $x - 3 \ne 0$

$\Rightarrow \quad 2 < x < 4$ and $x \ne 3$

$\Rightarrow \quad x \in (2, 4) - \{3\}$

Example 21 Solve: $\log_{0.5} \dfrac{3-x}{x+2} < 0$

Sol. $\log_{0.5} \dfrac{3-x}{x+2} < 0$

$\Rightarrow \quad \dfrac{3-x}{x+2} > (0.5)^0$

$\Rightarrow \quad \dfrac{3-x}{x+2} > 1$

$\Rightarrow \quad \dfrac{3-x}{x+2} - 1 > 0$

$\Rightarrow \quad \dfrac{3-x-x-2}{x+2} > 0$

$\Rightarrow \quad \dfrac{2x-1}{x+2} < 0$

$\Rightarrow \quad -2 < x < 1/2$

Example 22 Solve: $\log_{0.04}(x-1) \geq \log_{0.2}(x-1)$

Sol. $\log_{0.04}(x-1) \geq \log_{0.2}(x-1)$

$$\Rightarrow \quad \log_{(0.2^2)}(x-1) \geq \log_{0.2}(x-1)$$

$$\Rightarrow \quad \frac{1}{2}\log_{0.2}(x-1) \geq \log_{0.2}(x-1)$$

$$\Rightarrow \quad \log_{0.2}(x-1) \geq 2\log_{0.2}(x-1)$$

$$\Rightarrow \quad \log_{0.2}(x-1) \geq \log_{0.2}(x-1)^2$$

$\Rightarrow \quad (x-1) \leq (x-1)^2$

$\Rightarrow \quad (x-1)^2 - (x-1) \geq 0$

$\Rightarrow \quad (x-1)(x-1-1) \geq 0$

$\Rightarrow \quad (x-1)(x-2) \geq 0$

$\Rightarrow \quad x \leq 1$ or $x \geq 2$

Also $x > 1$

Hence $x \geq 2$

Example 23 Solve: $\log_3(x+2)(x+4) + \log_{1/3}(x+2) < \frac{1}{2}\log_{\sqrt{3}} 7$

Sol. $(x+2)(x+4) > 0, x+2 > 0$

$\Rightarrow \quad x > -2$

Now given inequality can be written as

$$\log_3(x+2)(x+4) - \log_3(x+2) < \frac{(\log 7)/2}{(\log 3)/2}$$

$\Rightarrow \quad \log_3(x+4) < \log_3 7$

$\Rightarrow \quad x+4 < 7$ or $x < 3$

Example 24 Solve: $\log_{1/2}(4-x) \geq \log_{1/2}2 - \log_{1/2}(x-1)$

Sol. $\log_{1/2}(4-x) \geq \log_{1/2}2 - \log_{1/2}(x-1)$

$\Rightarrow \quad \log_{1/2}(4-x)(x-1) \geq \log_{1/2}2$

$\Rightarrow \quad (4-x)(x-1) \leq 2$

$\Rightarrow \quad x^2 - 5x + 6 \geq 0$

$\Rightarrow \quad (x-3)(x-2) \geq 0$

$\Rightarrow \quad x \geq 3$ or $x \leq 2$

but $x \in (1, 4)$

$\Rightarrow \quad x \in (1, 2] \cup [3, 4)$

Example 25 Solve: $2 + \log_2\sqrt{x+1} > 1 - \log_{1/2}\sqrt{4-x^2}$

Sol. $2 + \log_2\sqrt{x+1} > 1 - \log_{1/2}\sqrt{4-x^2}$

$$\Rightarrow \quad 1 + \log_2 \sqrt{x+1} - \log_2 \sqrt{4-x^2} > 0$$

$$\Rightarrow \quad \log_2 2 + \log_2 \sqrt{x+1} - \log_2 \sqrt{4-x^2} > 0$$

$$\Rightarrow \quad \log_2 \frac{2\sqrt{x+1}}{\sqrt{4-x^2}} > 0$$

$$\Rightarrow \quad \frac{2\sqrt{x+1}}{\sqrt{4-x^2}} > 1$$

$$\Rightarrow \quad 4(x+1)^2 > 4 - x^2$$

$$\Rightarrow \quad 4x^2 + 8x + 4 > 4 - x^2$$

$$\Rightarrow \quad 5x^2 + 8x > 0$$

$$\Rightarrow \quad x > 0 \qquad \text{(i)}$$

Also $x + 1 > 0$ and $4 - x^2 > 0$

$$\Rightarrow \quad x > -1 \text{ and } -2 < x < 2 \qquad \text{(ii)}$$

From (i) and (ii), $0 < x < 2$

Example 26 Solve: $\log_9 (x+1) \cdot \log_2 (x+1) - \log_9 (x+1) - \log_2 (x+1) + 1 < 0$

Sol. We have $\log_9 (x+1) \cdot \log_2 (x+1) - \log_9 (x+1) - \log_2 (x+1) + 1 < 0$

$$\Rightarrow \quad \left(\log_9(x+1) - 1\right)\left(\log_2(x+1) - 1\right) < 0$$

$$\Rightarrow \quad \left(\log_9\left(\frac{x+1}{9}\right)\right)\left(\log_2\left(\frac{x+1}{2}\right)\right) < 0$$

Case I: $\log_9\left(\frac{x+1}{9}\right) > 0$

$$\Rightarrow \quad \frac{x+1}{9} > 1$$

$$\Rightarrow \quad x > 8. \qquad (1)$$

$$\therefore \quad \log_2\left(\frac{x+1}{2}\right) < 0$$

$$\Rightarrow \quad 0 < \frac{x+1}{2} < 1$$

$$\Rightarrow \quad -1 < x < 1 \qquad (2)$$

Form (1) and (2), we get $x \in \phi$.

Case II: $\log_9\left(\frac{x+1}{9}\right) < 0$

$\Rightarrow \quad 0 < \dfrac{x+1}{9} < 1$

$\Rightarrow \quad -1 < x < 8$ (3)

$\therefore \quad \log_2\left(\dfrac{x+1}{2}\right) > 0$

$\Rightarrow \quad \dfrac{x+1}{2} > 1$

$\Rightarrow \quad x > 1$ (4)

From (3) and (4), $x \in (1, 8)$.

So integral values of x are 2, 3, 4, 5, 6, 7.

Example 27 Solve: $\log_{(x+3)}(x^2 - x) < 1$

Sol. $\log_{(x+3)}(x^2 - x) < 1$

$x(x-1) > 0$

$\Rightarrow \quad x > 1$ or $x < 0$ (1)

Let $\quad x + 3 > 1$

$\Rightarrow \quad x > -2$

Here we have $x^2 - x < x + 3$

$\Rightarrow \quad x^2 - 2x - 3 < 0$

$\Rightarrow \quad (x-3)(x+1) < 0$

Hence $x \in (-1, 0) \cup (1, 3)$

Again, let $0 < x + 3 < 1$

$-3 < x < -2$ (1)

Then $\quad x^2 - x > x + 3$

$\Rightarrow \quad x^2 - 2x - 3 > 0$

$\Rightarrow \quad (x-3)(x+1) > 0$ (2)

Hence $x \in (-3, -2)$

Example 28 Solve: $\sqrt{\log_2 \dfrac{2x-3}{x-1}} < 1$

Sol. Inequality is true if

$$0 \le \log_2\left(\frac{2x-3}{x-1}\right) < 1$$

i.e. $\quad 1 \le \dfrac{2x-3}{x-1} < 2$

let, $\quad \dfrac{2x-3}{x-1} - 2 < 0 \Rightarrow \dfrac{2x-3-2x+2}{x-1} < 0$

$$\Rightarrow \quad \frac{-1}{x-1} < 0 \Rightarrow \frac{1}{x-1} > 0 \Rightarrow x > 1 \qquad (1)$$

and $\quad \dfrac{2x-3}{x-1} \geq 1 \Rightarrow \dfrac{2x-3}{x-1} - 1 \geq 0$

$$\Rightarrow \quad \frac{2x-3-x+1}{x-1} \geq 0 \Rightarrow \frac{x-2}{x-1} \geq 0 \Rightarrow x \geq 2 \text{ or } x < 1 \qquad (2)$$

Taking intersection of (1) and (2), $x \geq 2$.

Example 29 Solve: $\dfrac{1-\log_4 x}{1+\log_2 x} \leq \dfrac{1}{2}$

Sol. Let $\log_2 x = t$

$$\therefore \quad \frac{1-(t/2)}{1+t} \leq \frac{1}{2}$$

$$\Rightarrow \quad \frac{2-t}{1+t} \leq 1$$

$$\Rightarrow \quad \frac{2-t}{1+t} - 1 \leq 0$$

$$\Rightarrow \quad \frac{2t-1}{t+1} \geq 0$$

$$\Rightarrow \quad t < -1 \text{ or } t \geq \frac{1}{2}$$

$$\Rightarrow \quad \log_2 x < -1 \text{ or } \log_2 x \geq \frac{1}{2}$$

$$\Rightarrow \quad 0 < x < \frac{1}{2} \text{ or } x \geq \sqrt{2}$$

Example 30 Solve: $\log_{\frac{1}{x}}\left(\dfrac{2(x-2)}{(x+1)(x-5)}\right) \geq 1$

Sol. **Case I:** $1/x > 1$ or $0 < x < 1$

$$\therefore \quad \log_{\frac{1}{x}}\left(\frac{2(x-2)}{(x+1)(x-5)}\right) \geq 1$$

$$\Rightarrow \quad \frac{2(x-2)}{(x+1)(x-5)} \geq \frac{1}{x}$$

$$\Rightarrow \quad \frac{2(x-2)}{(x+1)(x-5)} - \frac{1}{x} \geq 0$$

$$\Rightarrow \qquad \frac{2x(x-2)-(x+1)(x-5)}{x(x+1)(x-5)} \geq 0$$

$$\Rightarrow \qquad \frac{x^2+5}{x(x+1)(x-5)} \geq 0$$

$\Rightarrow \qquad x \in (-1, 0) \cup (5, \infty)$

Hence no solution in this case

Case II: $0 < \dfrac{1}{x} < 1$ or $x > 1$

$\therefore \qquad 0 < x < 5$ (1)

Also $\dfrac{2(x-2)}{(x+1)(x-5)} > 0$

$\therefore \qquad x \in (1, 2)$

Example 31 Solve: $(x-2)^{x^2-6x+8} > 1$

Sol. Clearly, $x > 2$ (i)

$(x-2)^{x^2-6x+8} > 1$

$\Rightarrow \qquad (x-2)^{x^2-6x+8} > (x-2)^0$

Let $x - 2 > 1$

$\therefore \qquad x > 3$ (ii)

We have

$x^2 - 6x + 8 > 0$

$\Rightarrow \qquad (x-2)(x-4) > 0$

$\Rightarrow \qquad x < 2$ or $x > 4$ (iii)

From (ii) and (iii), $x > 4$

Let $x - 2 < 1$

$\therefore \qquad x < 3$ (iv)

So $x^2 - 6x + 8 < 0$

$\therefore \qquad (x-2)(x-4) < 0$

$\therefore \qquad 2 < x < 4$ (v)

From (iv) and (v);

$2 < x < 3$

Thus we have $x \in (2, 3) \cup (4, \infty)$

Example 32 Solve: $2(25)^x - 5(10^x) + 2(4^x) \geq 0$

Sol. $2(25)^x - 5(10^x) + 2(4^x) \geq 0$

$\Rightarrow \qquad 2(5)^{2x} - 5(5)^x(2^x) + 2(2^{2x}) \geq 0$

$$\Rightarrow \quad 2\left(\frac{5}{2}\right)^{2x} - 5\left(\frac{5}{2}\right)^{x} + 2 \geq 0$$

$$\Rightarrow \quad \left[\left(\frac{5}{2}\right)^{x} - 2\right]\left[2\left(\frac{5}{2}\right)^{x} - 1\right] \geq 0$$

$$\Rightarrow \quad \left(\frac{5}{2}\right)^{x} \leq \frac{1}{2} \text{ or } \left(\frac{5}{2}\right)^{x} \geq 2$$

$$\Rightarrow \quad x \leq \log_{2.5} 0.5 \text{ or } x \geq \log_{2.5} 2$$

Example 33 Solve: $\left(\frac{1}{2}\right)^{\log_{10} a^2} + 2 > \frac{3}{2^{\log_{10}(-a)}}$

Sol. $\left(\frac{1}{2}\right)^{\log_{10} a^2} + 2 > \frac{3}{2^{\log_{10}(-a)}}$

We must have $a < 0$

$$\Rightarrow \quad \frac{1}{2^{2\log(-a)}} + 2 > \frac{3}{2^{\log(-a)}}$$

Put $y = \frac{1}{2^{\log(-a)}}$

$\Rightarrow \quad y^2 + 2 > 3y$

$\Rightarrow \quad y^2 - 3y + 2 > 0$

$\Rightarrow \quad (y-1)(y-2) > 0$

$\Rightarrow \quad y < 1$ or $y > 2$

$\Rightarrow \quad 2^{-\log(-a)} < 1$ or $2^{-\log(-a)} > 2$

$\Rightarrow \quad -\log_{10}(-a) < 0$ or $-\log_{10}(-a) > 1$

$\Rightarrow \quad \log_{10}(-a) > 0$ or $\log_{10}(-a) < -1$

$\Rightarrow \quad -a > 1$ or $-a < 10^{-1}$

$\Rightarrow \quad a < -1$ or $a > -0.1$

$\Rightarrow \quad a \in (-\infty, -1) \cup (-0.1, \infty)$

Example 34 Solve: $(0.5)^{\log_3 \log_{(1/5)}\left(x^2 - \frac{4}{5}\right)} > 1$

Sol. $(0.5)^{\log_3 \log_{(1/5)}\left(x^2 - \frac{4}{5}\right)} > 1$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^{\log_3 \log_{\frac{1}{5}}\left(x^2 - \frac{4}{5}\right)} > \left(\frac{1}{2}\right)^{0}$$

$$\Rightarrow \quad \log_3 \log_{\frac{1}{5}}\left(x^2 - \frac{4}{5}\right) < 0$$

$$\Rightarrow \quad 0 < \log_{\frac{1}{5}}\left(x^2 - \frac{4}{5}\right) < 1$$

$$\Rightarrow \quad 1 > x^2 - \frac{4}{5} > \frac{1}{5}$$

$$\Rightarrow \quad 1 < x^2 < \frac{9}{5}$$

$$\Rightarrow \quad x \in \left(-\frac{3}{5}, -1\right) \cup \left(1, \frac{3}{5}\right)$$

Example 35 Find the values of x for which the function $f(x) = \sqrt{\log_{\frac{1}{2}} \frac{x-1}{x+5}}$ is defined.

Sol. $f(x) = \sqrt{\log_{\frac{1}{2}} \frac{x-1}{x+5}}$

It is defined if $\log_{\frac{1}{2}} \frac{x-1}{x+5} \geq 0$

$$\Rightarrow \quad 0 < \frac{x-1}{x+5} \leq 1$$

For $\frac{x-1}{x+5} > 0$, $x \in (-\infty, -5) \cup (1, \infty)$ ...(i)

For $\frac{x-1}{x+5} \leq 1$

$$\frac{x-1}{x+5} - 1 \leq 0$$

$$\Rightarrow \quad \frac{-6}{x+5} \leq 0$$

$$\Rightarrow \quad x > -5 \qquad \text{...(ii)}$$

From (i) and (ii), $x \in (1, \infty)$.

Exercise 1

1. Solve: $1 < \log_2(x-2) \leq 2$

2. Solve: $\log_2|x-1| < 1$
3. Solve: $\log_3|x| > 2$
4. Solve: $\log_2|4-5x| > 2$
5. Solve: $\log_2 \dfrac{x-1}{x-2} > 0$
6. Solve: $\log_2 \dfrac{x-4}{2x+5} < 1$
7. Solve: $\log_{0.2} \dfrac{x+2}{x} \le 1$
8. Solve: $\log_3(2x^2+6x-5) > 1$
9. Solve: $\log_{1/2}(x^2-6x+12) \ge -2$
10. Solve: $2\log_3 x - 4\log_x 27 \le 5$ $(x > 1)$
11. Let $f(x) = \sqrt{\log_{10} x^2}$. Then find the set of all values of x for which $f(x)$ is real.
12. Solve: $2^{\log_2(x-1)} > x+5$
13. Solve: $x^{\log_5 x} > 5$
14. Solve: $(\log_{0.6} 0.216)\log_5(5-2x) \le 0$
15. Solve: $\log_3(x+2)(x+4) + \log_{1/3}(x+2) < \frac{1}{2}\log_{\sqrt{3}} 7$
16. Solve: $\log_{10}(x^2-16) \le \log_{10}(4x-11)$
17. Solve: $\log_{\sqrt{0.9}} \log_5(\sqrt{x^2+5}+x) > 0$
18. Solve: $\log_{1/2}(4-x) \ge \log_{1/2}2 - \log_{1/2}(x-1)$
19. Solve: $2^{x+2} - 2^{x+3} - 2^{x+4} > 5^{x+1} - 5^{x+2}$
20. Solve: $\left(\dfrac{3}{4}\right)^{6x+10-x^2} < \dfrac{27}{64}$

Exercise 2

1. Solve: $\log_{1-x}(x-2) \ge -1$
2. Solve: $\log_2\sqrt{x} - 2(\log_{1/4} x)^2 + 1 > 0$
3. Solve: $2\log_{\frac{1}{4}}(x+5) > \dfrac{9}{4}\log_{\frac{1}{3\sqrt{3}}}(9) + \log_{\sqrt{x+5}}(2)$

4. Solve: $\log_3 x - (\log_3 x)^2 \le \frac{3}{2}\log_{(1/2\sqrt{2})}4$

5. Solve: $\log_x (x^2 - 1) \le 0$

6. Solve: $\log_{0.8}\left(\log_6 \frac{x^2+x}{x+4}\right) < 0$

7. Solve: $\log_3 (x^2 - 2) < \log_3 \left(\frac{3}{2}|x| - 1\right)$

8. Solve: $\log_{(x+1)}(x^2 - 4) > 1$

9. Find the domain of function $f(x) = \log_4 \{\log_5(\log_3 (18x - x^2 - 77))\}$

10. Solve: $\frac{x-1}{\log_3(9-3^x)-3} \le 1$

11. Solve: $\frac{16^{\frac{1}{x}}}{2^{x+3}} > 1$

12. Solve: $2(25)^x - 5(10^x) + 2(4^x) \ge 0$

CHAPTER

6

Using Logarithmic Table

Finding Logarithm

To calculate the logarithm of any positive number in decimal form, we always express the given positive number in the decimal form as the product of an integral power of 10 and a number between 1 and 10, i.e., any positive number n in the decimal form is written as

$$n = m \times 10^p$$

where p is an integer and $1 \leq m < 10$. This is called the standard form of k.

Characteristic and Mantissa of a Logarithm

Let n be a positive real number and $m \times 10^p$ be the standard form of n. Then $n = m \times 10^p$, where p is an integer and m is a real number between 1 and 10, i.e., $1 \leq m < 10$. Thus,

$$\begin{aligned} \log_{10} n &= \log_{10} (m \times 10^p) \\ &= \log_{10} m + \log_{10} 10^p \\ &= \log_{10} m + p \log_{10} 10 \\ &= p + \log_{10} m \end{aligned}$$

Thus,

$$\log_{10} 1 \leq \log_{10} m < \log_{10} 10$$

$$\Rightarrow \quad 0 \leq \log_{10} m < 1$$

Thus, the logarithm of a positive real number n consists of two parts:

i. The integral part p, called characteristic, which is positive, negative, or zero.

ii. The decimal part $\log m$, called mantissa, which is a real number between 0 and 1.

Thus, $\log n$ = Characteristic + Mantissa

Note that it is only the characteristic that changes when the decimal point is moved. An advantage of using the base 10 is thus revealed: if the characteristic is known, the decimal point may easily be placed. If the number is known, the characteristic may be determined by inspection, i.e., by observing the location of the decimal point.

Although an understanding of the relation of the characteristic to the powers of 10 is necessary for a thorough comprehension of logarithms, the characteristic may be determined mechanically by the application of the following rules:

1. For a number greater than 1, the characteristic is positive and is one less than the number of digits to the left of the decimal point in the number.
2. For a positive number less than 1, the characteristic is negative and has an absolute value one more than the number of zeros between the decimal point and the first nonzero digit of the number.

Example 1 Write the characteristic of each of the following numbers by using their standard forms:

i. 1235.5	**ii.** 346.41
iii. 62.723	**iv.** 7.12345
v. 0.35792	**vi.** 0.034239
vii. 0.002385	**viii.** 0.0009468

Sol.

Number	Standard form	Characteristic
1235.5	1.2355×10^3	3
346.41	3.4641×10^2	2
62.723	6.2723×10^1	1
7.12345	7.12345×10^0	0
0.35792	3.5792×10^{-1}	–1
0.034239	3.4239×10^{-2}	–2
0.002385	2.385×10^{-3}	–3
0.0009468	9.468×10^{-4}	–4

Mantissa of the Logarithm of a Given Number

The logarithm table is used to find the mantissa of logarithms of numbers. It contains 90 rows and 20 columns. Every row beings with a two-digit number 10, 11, 12, ..., 98, 99 and every column is headed by a one-digit number 0, 1, 2, 3, ..., 9. On the right of the table, we have a big column which is divided into nine sub-columns headed by the digits 1, 2, 3, ..., 9. This column is called the column of mean differences.

Note that the position of the decimal point in a number is immaterial for finding the mantissa. To find the mantissa of a number, we consider the first four digits from the left most side of the number. If the number in the decimal form is less than one and it has four or more consecutive zeros to the right of the decimal point, then

its mantissa is calculated with the help of the number formed by digits beginning with the first nonzero digit. For example, to find the mantissa of 0.000032059, we consider the number 3205. If the given number has only one digit, we replace it by a two-digit number obtained by adjoining zero to the right of the number. Thus, 2 is to be replaced by 20 for finding the mantissa.

Significant Digits

The digits used to compute the mantissa of a given number are called its significant digits.

Example 2 Write the significant digits in each of the following numbers to compute the mantissa of their logarithms :

i. 3.239	**ii.** 8	**iii.** 0.9	**iv.** 0.02
v. 0.0367	**vi.** 89	**vii.** 0.0003	**viii.** 0.00075

Sol.

Number	Significant digits to find the mantissa of its logarithm
3.239	3239
8	80
0.9	90
0.02	20
0.0367	367
89	89
0.0003	30
0.00075	75

Negative Characteristics

When a characteristic is negative, such as –2, we do not perform the subtraction, because this would involve a negative mantissa. There are several ways of indicating a negative characteristic. Mantissas as presented in the table in the appendix are always positive, and the sign of the characteristic is indicated separately. For example, consider log 0.023 = $\bar{2}.36173$. Here the bar over 2 indicates that only the characteristic is negative, i.e., the logarithm is $-2 + 0.36173$.

Example 3 Find the mantissa of the logarithm of the number 5395.

Sol. To find the mantissa of log 5395, we first look into the row starting with 53. In this row, look at the number in the column headed by 9. The number is 7316.

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

Fig. 6.1

Now, move to the column of mean differences and look under the column headed by 5 in the row corresponding to 53. We see the number 4 there. Add this number 4 to 7316 to get 7320. This is the required mantissa of log 5395.

If we wish to find log 5395, then we compute its characteristic also.

Clearly, the characteristic is 3. So, log 5395 = 3.7320.

Example 4 Find the mantissa of the logarithm of the number 0.002359.

Sol. The first four digits beginning with the first nonzero digit on the right of the decimal point form the number 2359. To find the mantissa of log (0.002359), we first look in the row starting with 23. In this row, look at the number in the column headed by 5. The number is 3711.

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	15	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15

Fig. 6.2

Now, move to the column of mean differences and look under the column headed by 9 in the row corresponding to 23. We see the number 17 there.

Add this number to 3711. We get the number 3728. This is the required mantissa of log (0.002359).

Mantissa of log 23.598, log 2.3598, and 0.023598 is the same (only characteristics are different).

Example 5 Use logarithm tables to find the logarithm of the following numbers:
i. 25795 **ii.** 25.795

Sol.

i. The characteristic of the logarithm of 25795 is 4.
To find the mantissa of the logarithm of 25795, we take the first four digits. The number formed by the first four digits is 2579. Now, we look in the row starting with 25. In this row, look at the number in the column headed by 7. The number is 4099. Now, move to the column of mean differences and look under the column headed by 9 in the row corresponding to 25. We see that the number there is 15.
Add this number to 4099. We get the number 4114. This is the required mantissa. Hence,
$$\log (25795) = 4.4114$$

ii. The characteristic of the logarithm of 25.795 is 1, because there are two digits to the left of the decimal point. The mantissa is the same as in the above question. Hence,
$$\log 25.795 = 1.4114.$$
Similarly, $\log 2.5795 = 0.4114.$
and $\log (0.25795) = -1 + 0.4114 = \bar{1}.4114$
Here $-1 + 04114$ cannot be written as -1.4114, as -1.4114 is a negative number of magnitude 1.4114, whereas $-1 + 0.4114$ is equal to -05886. In order to avoid this confusion, we write $\bar{1}$ for -1; thus.
$$\log(0.25795) = \bar{1}.4114.$$

ANTILOGARITHM

The positive number n is called the antilogarithm of a number m if $\log n = m$. If n is antilogarithm of m, we write $n = \text{antilog } m$. For example,

i. $\log 100 = 2 \Leftrightarrow \text{antilog } 2 = 100$

ii. $\log 431.5 = 2.6350 \Leftrightarrow \text{antilog } (2.6350) = 431.5$

iii. $\log 0.1257 = \bar{1}.993 \Leftrightarrow \text{antilog } (\bar{1}.993) = 0.1257$

To find the antilog of a given number, we use the antilogarithm tables given as an appendix at the end of the book. To find n, when $\log n$ is given, we use only the mantissa part. The characteristic is used only in determining the number of digits in the integral part or the number of zeros on the right side of the decimal point in the required number.

To Find Antilog of a Number

Step I: Determine whether the decimal part of the given number is positive or negative. If it is negative, make it positive by adding 1 to the decimal part and by subtracting 1 from the integral part.
For example, in -2.5983, the decimal part is -0.5983 which is negative. So, write

$$-2.5983 = -2 - 0.5983$$
$$= -2 - 1 + 1 - 0.5983$$
$$= -3 + 0.4017$$
$$= \bar{3}.4017$$

Step II: In the antilogarithm table, look into the row containing the first two digits in the decimal part of the given number.

Step III: In the row obtained in step II, look at the number in the column headed by the third digit in the decimal part.

Step IV: In the row chosen in step III, move in the column of mean differences and look at the number in the column headed by the fourth digit in the decimal part. Add this number to the number obtained in step III.

Step V: Obtain the integral part (characteristic) of the given number.
If the characteristic is positive and is equal to n, then insert a decimal point after $(n + 1)$ digits in the number obtained in step IV.
If $n > 4$, then write zeros on the right side to get $(n + 1)$ digits.
If the characteristic is negative and is equal to $-n$ or n, then on the right side of decimal point write $(n - 1)$ consecutive zeros and then write the number obtained in step IV.

Example 6 Find the antilogarithm of each of the following:

i. 2.7523 **ii.** 3.7523
iii. 5.7523 **iv.** 0.7523
v. $\bar{1}.7523$ **vi.** $\bar{2}.7523$
vii. $\bar{3}.7523$

Sol.

i. The mantissa of 2.7523 is positive and is equal to 0.7523.
Now, look into the row starting 0.75. In this row, look at the number in the column headed by 2. The number is 5649. Now in the same row move in the column of mean differences and look at the number in the column headed by 3. The number there is 4. Add this number to 5649 to get 5653.
The characteristic is 2. So, the decimal point is put after three digits to get 565.3. Hence,
$$\text{antilog}(2.7523) = 565.3$$

ii. The mantissa of 3.7523 is the same as the mantissa of the number in Step (i), but the characteristic is 3. Hence,
$$\text{antilog}(3.7523) = 5653.0$$

iii. The mantissa of 5.7523 is the same as the mantissa of the number in Step (i), but the characteristic is 5. Hence,
$$\text{antilog}(5.7523) = 565300.0.$$

iv. Proceeding as above, we have antilog (0.7523) = 5.653.

v. In this case, the characteristic is $\bar{1}$, i.e., -1. Hence,
$$\text{antilog}\ (\bar{1}.7523) = 0.5653$$

vi. In this case, the characteristic is $\bar{2}$, i.e., -2. So, we write one zero on the right side of the decimal point. Hence,

$$\text{antilog}\,(\bar{2}.7523) = 0.05653$$

vii. Proceeding as above, $\text{antilog}(\bar{3}.7523) = 0.005653$.

Example 7 Evaluate $\sqrt[3]{72.3}$, if $\log 0.723 = \bar{1}.8591$.

Sol. Let $x = \sqrt[3]{72.3}$. Then

$$\log x = \log(72.3)^{1/3} \text{ or } \log x = \frac{1}{3}\log 72.3$$

or $$\log x = \frac{1}{3} \times 1.8591 \Rightarrow \log x = 0.6197$$

or $$x = \text{antilog}(0.6197)$$
$$= 4.166 \qquad \text{(using antilog table)}$$

Example 8 Using logarithms, find the value of 6.45×981.4.

Sol. Let $x = 6.45 \times 981.4$. Then,

$$\log x = \log(6.45 \times 981.4)$$
$$= \log 6.45 + \log 981.4$$
$$= 0.8096 + 2.9919 \text{ (using log table)}$$
$$= 3.8015$$
$$\therefore \quad x = \text{antilog}(3.8015) = 6331$$

(using antilog table)

Example 9 Let $x = (0.15)^{20}$. Find the characteristic and mantissa of the logarithm of x to the base 10. Assume $\log_{10}2 = 0.301$ and $\log_{10}3 = 0.477$.

Sol. $\log x = \log(0.15)^{20} = 20\log\left(\frac{15}{100}\right)$

$$= 20[\log 15 - 2]$$
$$= 20[\log 3 + \log 5 - 2]$$
$$= 20[\log 3 + 1 - \log 2 - 2] \qquad \left(\because \log_{10} 5 = \log_{10}\frac{10}{2}\right)$$
$$= 20[-1 + \log 3 - \log 2]$$
$$= 20[-1 + 0.477 - 0.301]$$
$$= -20 \times 0.824 = -16.48 = \overline{17}.52$$

Hence, Characteristic $= -17$ and Mantissa $= 0.52$

Example 10 If $\log_{10} 2 = 0.30103$, $\log_{10} 3 = 0.47712$, then find the number of digits in $3^{12} \times 2^8$.

Sol. Let $y = 3^{12} \times 2^8$

$$\Rightarrow \quad \log_{10} y = 12\log_{10} 3 + 8\log_{10} 2$$
$$= 12 \times 0.47712 + 8 \times 0.30103$$

$= 5.72544 + 2.40824$

$= 8.13368$

$\therefore$ Number of digits in $y = 8 + 1 = 9$.

Example 11 If $\log_2(\log_2(\log_2 x)) = 2$ then find the number of digits in x ($\log_{10}2 = 0.3010$).

Sol. $\log_2(\log_2(\log_2 x)) = 2$

$$\log_2(\log_2 x) = 4$$

$$\Rightarrow \quad \log_2 x = 16$$

$$\Rightarrow \quad x = 2^{16}$$

$$\therefore \quad \log_{10}x = 16 \log_{10}2 = 16 \times 0.3010 = 4.8160$$

$\therefore$ Number of digits in $x = 5$

Example 12 Find the number of digits in $8^{12}\,5^{35}$.

Sol. Let $N_1 = 8^{12}\,5^{35}$

$$\Rightarrow \quad \log_{10} N_1 = \log_{10}(2^{36} \cdot 5^{35})$$

$$= \log_{10}(10^{35} \cdot 2)$$

$$= 35 + \log_{10} 2$$

$$= 35 + 0.3010$$

$$= 35.3010$$

Thus, number of digits in N_1 is 36.

Example 13 Find the number of zeroes after decimal before a significant figure start in $\left(\frac{8}{27}\right)^{20}$.

Sol. Now, let $N_2 = \left(\frac{8}{27}\right)^{20}$

$$\Rightarrow \quad \log_{10} N_2 = 20 \log_{10}\left(\frac{8}{27}\right)$$

$$= 60 \log_{10}\frac{2}{3}$$

$$= 60(\log_{10} 2 - \log_{10}3)$$

$$= 60 \times (0.3010 - 0.4771)$$

$$= -60 \times 0.1761$$

$$= -10.56$$

$$= \overline{11} + 0.44$$

Therefore, number of zeroes after decimal before a significant figure start in N_2 is 10.

Example 14 If $\log_{10} 2 = 0.30103$ and $\log_{10} 3 = 0.47712$, then find the number of digits in $3^{12} \times 2^8$.

Sol. Let $y = 3^{12} \times 2^8$

$$\Rightarrow \log_{10} y = 12\log_{10} 3 + 8\log_{10} 2$$
$$= 12 \times 0.47712 + 8 \times 0.30103$$
$$= 5.72544 + 2.40824$$
$$= 8.13368$$

$\therefore$ Number of digits in $y = 8 + 1 = 9$.

Example 15 In the 2001 census, the population of India was found to be 8.7×10^7. If the population increases at the rate of 2.5% every year, what would be the population in 2011?

Sol. Here, $P_0 = 8.7 \times 10^7$, $r = 2.5$, and $n = 10$.

Let P be the population in 2011. Then,

$$P = P_0\left(1 + \frac{r}{100}\right)^n$$
$$= 8.7 \times 10^7\left(1 + \frac{2.5}{100}\right)^{10}$$
$$= 8.7 \times 10^7 (1.025)^{10}$$

Taking log of both sides, we get

$$\log P = \log [\, 8.7 \times 10^7(1.025)^{10}]$$
$$= \log 8.7 + \log 10^7 + \log (1.025)^{10}$$
$$= \log 8.7 + 7 \log 10 + 10 \log (1.025)$$
$$= 0.9395 + 7 + 0.1070$$
$$= 8.0465$$
$$\Rightarrow \quad P = \text{antilog}(8.0465) = 1.113 \times 10^8$$

(using antilog table)

Example 16 Find the compound interest on ₹ 12000 for 10 years at the rate of 12% per annum compounded annually.

Sol. We know that the amount A at the end of n years at the rate of r% per annum when the interest is compounded annually is given by

$$A = P\left(1 + \frac{r}{100}\right)^n$$

Here, $P = ₹\ 12000$, $r = 12$, and $n = 10$.

$$\therefore \quad A = ₹\left[12000\left(1 + \frac{12}{100}\right)^{10}\right]$$
$$= ₹\left[12000\left(1 + \frac{3}{25}\right)^{10}\right]$$

$$= ₹ \left[12000\left(\frac{25+3}{25}\right)^{10}\right]$$

$$= ₹ \left[12000\left(\frac{28}{25}\right)^{10}\right]$$

Now, $A = ₹\ 12000\left(\frac{28}{25}\right)^{10}$

$\Rightarrow$ $\log A = \log 12000 + 10 (\log 28 - \log 25)$

$= 4.0792 + 10 (1.4472 - 1.3979)$

$= 4.0792 + 0.493 = 4.5722$

$\Rightarrow$ $A = \text{antilog}(4.5722) = 37350.$

So, the amount after 10 years is ₹ 37350.

Hence, Compound interest = ₹ (37350 – 12000) = ₹ 25350

Example 17 If P is the number of natural numbers whose logarithms to the base 10 have the characteristic p and Q is the number of natural numbers logarithms of whose reciprocals to the base 10 have the characteristic $-q$, then find the value of $\log_{10}P - \log_{10}Q$.

Sol. $10^p \leq P < 10^{p+1} \Rightarrow P = 10^{p+1} - 10^p \Rightarrow P = 9 \times 10^p$

Similarly, $10^{q-1} < Q \leq 10^q$

$\Rightarrow$ $Q = 10^q - 10^{q-1} = 10^{q-1}(10 - 1) = 9 \times 10^{q-1}$

$\therefore$ $\log_{10}P - \log_{10}Q = \log_{10}(P/Q) = \log_{10}10^{p-q+1}$

$= p - q + 1$

Example 18 Let L denote $\text{antilog}_{32} 0.6$ and M denote the number of positive integers which have the characteristic 4, when the base of log is 5, and N denote the value of $49^{(1-\log_7 2)} + 5^{-\log_5 4}$. Find the value of LM/N.

Sol. $L = \text{antilog}_{32} 0.6 = (32)^{6/10} = 2^{(5\times6)/10} = 2^3 = 8$

M = Integer from 625 to 3125 = 2500

$N = 49^{(1-\log_7 2)} + 5^{-\log_5 4}$

$= 49 \times 7^{-2\log_7 2} + 5^{-\log_5 4}$

$= 49 \times \frac{1}{4} + \frac{1}{4} = \frac{50}{4} = \frac{25}{2}$

$\therefore$ $\frac{LM}{N} = \frac{8 \times 2500 \times 2}{25} = 1600$

Appendix

1 Logarithm and Antilogarithm Tables

TABLE A-1.1 Logarithms

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						5	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	18	22	26	29	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	32
13	1139	1173	1206	1239	1271						3	6	10	13	16	19	23	26	29
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	14	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	6	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	3	5	8	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	17	19	21
						2672	2695	2718	2742	2765	2	4	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3118	3201	2	4	6	8	11	13	13	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17

A.2 Logarithm and its Applications

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6069	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6014	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

TABLE A-1.2 Antilogarithms

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4557	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Appendix

2 Hints and Solutions

Chapter 1

Exercise 1

1. (i) $\log_{10}0.001 = x$

$\therefore \quad 0.001 = 10^x$

$\therefore \quad 10^{-3} = 10^x$

$\therefore \quad x = -3$

(ii) $\log_2(1/32) = x$

$\therefore \quad 1/32 = 2^x$

$\therefore \quad 2^{-5} = 2^x$

$\therefore \quad x = -5$

(iii) $x = 0.\overline{1} = 0.1111111\ldots$

$\therefore \quad 10x = 1.1111111\ldots$

Subtracting we get $9x = 1$

$\therefore \quad x = 1/9$

Now $\log_{9\sqrt{3}} 0.\overline{1} = \log_{9\sqrt{3}} \dfrac{1}{9} = x$

$\therefore \quad \dfrac{1}{9} = (9\sqrt{3})^x$

$\therefore \quad 3^{-2} = 3^{2.5x}$

$\therefore \quad 2.5x = -2$

$\therefore \quad x = -2/2.5 = -4/5$

(iv) $\log_{(5+2\sqrt{6})}(5-2\sqrt{6}) = x$

$\therefore \quad (5-2\sqrt{6}) = (5+2\sqrt{6})^x$

$$\therefore \quad \frac{(5-2\sqrt{6})(5+2\sqrt{6})}{(5+2\sqrt{6})} = (5+2\sqrt{6})^x$$

$$\therefore \quad \frac{1}{(5+2\sqrt{6})} = (5+2\sqrt{6})^x$$

$$\therefore \quad (5+2\sqrt{6})^{-1} = (5+2\sqrt{6})^x$$

$$\therefore \quad x = -1$$

2. (i) $\log_{81} x = \dfrac{3}{2}$

$$\therefore \quad x = (81)^{1.5} = (9^2)^{1.5} = 9^3 = 729$$

(ii) $\log_{\sqrt{2}} x = 4$

$$\therefore \quad x = (\sqrt{2})^4 = 4$$

(iii) $\log_2 4x = 5$

$$\therefore \quad 4x = 2^5$$

$$\therefore \quad x = 2^3 = 8$$

3. (i) $\log_{1/4}\left(\dfrac{1}{16}\right)^{-2} = \log_{1/4}\left(\dfrac{1}{4}\right)^{-4} = -4$

(ii) $\sqrt{(\log_{0.5} 4)^2}$

$= |\log_{0.5} 4|$

$= |\log_{0.5}(0.5)^{-2}|$

$= |-2|$

$= 2$

(iii) $\dfrac{\log_2 32}{\log_3 \sqrt{243}}$

$$= \frac{\log_2(2^5)}{\log_3(3^{5/2})}$$

$$= \frac{5}{5/2} = 2$$

4. $\log_3\left(\tan\dfrac{7\pi}{6}\right) + \log_{\cot(4\pi/3)}(3)$

$$= \log_3\left(\frac{1}{\sqrt{3}}\right) + \log_{1/\sqrt{3}}(3)$$

$= \log_3 3^{-1/2} + \log_{1/\sqrt{3}} (1/\sqrt{3})^{-2}$

$= -1/2 - 2$

$= -5/2$

5. $\log_2 \log_2 \log_4 256 + 2\log_{\sqrt{2}} 2$

$= \log_2 \log_2 \log_4 4^4 + 2\log_{\sqrt{2}} (\sqrt{2})^2$

$= \log_2 \log_2 4 + 4$

$= \log_2 2 + 4$

$= 1 + 4 = 5$

6. $\log_{\sqrt{8}} b = 3\frac{1}{3}$

$$\Rightarrow \quad b = \left(\sqrt{8}\right)^{\frac{10}{3}} = \left(2^{\frac{3}{2}}\right)^{\frac{10}{3}} = 2^5 = 32$$

7. log tan 1° log tan 2° … log tan 89°

= log tan 1° log tan 2° … log tan 45° … log tan 89°

= log tan 1° log tan 2° … (log 1) … log tan 89°

= 0 (as log 1 = 0)

8. If possible, let $\log_4 18 = \frac{p}{q}$ where $p, q \in I$

$$\Rightarrow \quad \log_4 9 + \log_4 2 = \frac{p}{q} \Rightarrow \frac{1}{2}.2\log_2 3 + \frac{1}{2} = \frac{p}{q}$$

$$\Rightarrow \quad \log_2 3 = \frac{p}{q} - \frac{1}{2} = \frac{m}{n}(say)$$

where $m, n \in I$ and $n \neq 0 \Rightarrow 3 = (2)^{m/n} \Rightarrow 3^n = 2^m$ (possible only when $m = n = 0$ which is not true)

Hence $\log_4 18$ is an irrational number.

9. (a)

(a) $\log_{10}\pi \cong 1/2$

(b) $\sqrt{\log_{10} \pi^2} \cong 1$

(c) $\left(\frac{1}{\log_{10} \pi}\right)^3 \cong 8$

(d) $\left(\frac{1}{\log_{10} \sqrt{\pi}}\right) \cong 4$

10. (i) +ve; (ii) –ve; (iii) +ve; (iv) +ve; (v) + ve; (vi) –ve; (vii) +ve; (viii) –ve

11. $\log_5 x = a$

$\therefore \quad x = 5^a$

$\log_2 y = a$

$\therefore \quad y = 2^a$

$$100^{2a-1} = (5^2 2^2)^{2a} \times 100^{-1}$$

$$= (5^a 2^a)^4 \times 100^{-1}$$

$$= \frac{(xy)^4}{100}$$

12. $\log_x \log_{18}\left(\sqrt{2} + 2\sqrt{2}\right) = -1/2$

$\Rightarrow \quad \log_x \log_{18} \sqrt{18} = -1/2$

$\Rightarrow \quad \log_x (1/2) = -1/2$

$\Rightarrow \quad 1/2 = x^{-1/2}$

$\Rightarrow \quad 4^{-1/2} = x^{-1/2}$

$\Rightarrow \quad x = 4$

Exercise 2

1. $3^{-2} 3^{\log_{\sqrt{5}} x} = 3^{-2}$

$\Rightarrow \quad 3^{\log_{\sqrt{5}} x} = 1$

$\Rightarrow \quad \log_{\sqrt{5}} x = 0$

$\Rightarrow \quad x = 1$

2. Let $x = \frac{1}{2\sqrt{3}}\sqrt{6 - \frac{1}{2\sqrt{3}}\sqrt{6 - \frac{1}{2\sqrt{3}}\sqrt{6 - \frac{1}{2\sqrt{3}}\ldots\infty}}}$

$\Rightarrow \quad x = \frac{1}{2\sqrt{3}}\sqrt{6 - x}$

$\Rightarrow \quad 12x^2 + x - 6 = 0$

$\Rightarrow \quad (3x - 2)(4x + 3) = 0$

$\Rightarrow \quad x = 2/3$

$\Rightarrow \quad \log_{\frac{9}{4}}\left(\frac{1}{2\sqrt{3}}\sqrt{6 - \frac{1}{2\sqrt{3}}\sqrt{6 - \frac{1}{2\sqrt{3}}\sqrt{6 - \frac{1}{2\sqrt{3}}\ldots\infty}}}\right)$

$$= \log_{\frac{9}{4}}\left(\frac{2}{3}\right) = -\frac{1}{2}$$

3. $\log_2\left(2\sqrt[3]{9}-2\right)\left(12\sqrt[3]{3}+4+4\sqrt[3]{9}\right)$

$$= \log_2\left((72)^{1/3}-(8)^{1/3}\right)\left((72)^{2/3}+(8)^{2/3}+(72)^{1/3}(8)^{1/3}\right)$$

$$= \log_2((72)-8)$$

$$= \log_2 64 = 6$$

4. Here, $5 = 4^a$ and $6 = 5^b$.

Let $\log_3 2 = x$. Then $2 = 3^x$.

Now, $6 = 5^b = (4^a)^b = 4^{ab}$ or $3 = 2^{2ab-1}$

$\therefore \quad 2 = (2^{2ab-1})^x = 2^{x(2ab-1)}$

$\Rightarrow \quad x(2ab-1) = 1$.

5. $\log_2\left(\log_2(\log_3 x)\right) = 0$

$\Rightarrow \quad \log_2(\log_3 x) = 1$

$\Rightarrow \quad \log_3 x = 2$

$\Rightarrow \quad x = 9$

Similarly, $\log_2\left(\log_3(\log_2 y)\right)$

$\Rightarrow \quad \log_3(\log_2 y) = 1$

$\Rightarrow \quad \log_2 y = 3$

$\Rightarrow \quad y = 8$

$\therefore \quad x - y = 1$

6. Put $\log_{175}5x = \log_{343}7x = k$

$\Rightarrow \quad 5x = 175^k$ and $7x = 343^k$

$$\Rightarrow \quad \frac{5}{7} = \left(\frac{175}{343}\right)^k$$

$$\Rightarrow \quad k = \frac{1}{2}$$

Now, $5x = (175)^{1/2}$

$\Rightarrow \quad x = \sqrt{7}$

$\Rightarrow \quad x^4 - 2x^2 + 7 = 42$

$\Rightarrow \quad \log_{42}(x^4 - 2x^2 + 7) = 1$

7. Let $\log_4 A = \log_6 B = \log_9(A + B) = x$

$\Rightarrow \quad A = 4^x, B = 6^x$ and $A + B = 9^x$

$\Rightarrow \quad 4^x + 6^x = 9^x$

$\Rightarrow \quad 2^{2x} + 2^x \cdot 3^x = 3^{2x}$

$\Rightarrow \quad (3/2)^{2x} - (3/2)^{x} - 1 = 0$

$\Rightarrow \quad \left(\frac{3}{2}\right)^x = \frac{\sqrt{5}+1}{2}$

8. $10^x + 10^{-x} = 4$

$\Rightarrow \quad 10^{2x} - 4 \cdot 10^x + 1 = 0$

$\Rightarrow \quad 10^x = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2+\sqrt{3}$ or $2-\sqrt{3}$

9. $\log_b n = 2$

$\Rightarrow \quad n = b^2$

$\log_n 2b = 2$

$\Rightarrow \quad 2b = n^2$

$\Rightarrow \quad 2 = b^3$

$\Rightarrow \quad b = 2^{1/3}$

10. $\frac{\log_2 x}{4} = \frac{\log_2 y}{6} = \frac{\log_2 z}{3k} = \lambda$

$\log_2 x = 4\lambda$, $\log_2 y = 6\lambda$, $\log_2 z = 3k\lambda$

$\therefore \quad x = 2^{4\lambda}, y = 2^{6\lambda}, z = 2^{3k\lambda}$

Given $x^3y^2z = 1$

$\therefore \quad 2^{12\lambda}2^{12\lambda}2^{3k\lambda} = 1$

$\therefore \quad 24\lambda + 3k\lambda = 0$

$\therefore \quad k = -8$

11. $\log_{10}(x+y) = z$

$\Rightarrow \quad x + y = 10^z$ (1)

And $x^2 + y^2 = 10 \cdot 10^z$ (2)

Squaring (1), we get $(x+y)^2 = 10^{2z}$

$\Rightarrow \quad x^2 + y^2 + 2xy = 10^{2z}$

$\Rightarrow \quad 2xy = 10^{2z} - 10 \cdot 10^z$ (3)

Now, $x^3 + y^3 = (x+y)^3 - 3xy(x+y)$

$= (10)^{3z} - \frac{3}{2}\{(10)^{2z} - 10 \cdot (10)^z\}10^z$

$= (10)^{3z} - \frac{3}{2}(10)^{3z} + 15 \cdot (10)^{2z}$

$= -\frac{1}{2}(10)^{3z} + 15(10^{2z})$

$\therefore \quad a = -\frac{1}{2}$ and $b = 15$

$\Rightarrow \quad a + b = \frac{29}{2}$

Chapter 2

Exercise 1

1. (i) $\log_{10}5 + \log_{10}2$

$$= \log_{10}5 \times 2$$
$$= \log_{10}10$$
$$= 1$$

(ii) $\frac{1}{2}\log_{10}36 + \log_{10}5 - \log_{10}30$

$$= \log_{10}36^{1/2} + \log_{10}5 - \log_{10}30$$
$$= \log_{10}6 + \log 5 - \log 30$$
$$= \log_{10}(6 \times 5) - \log_{10}30$$
$$= \log_{10}30 - \log_{10}30$$
$$= 0$$

(iii) $\log_{10}5 + 2\log_{10}0.5 + 3\log_{10}2$

$$= \log_{10}5 + \log_{10}0.25 + \log_{10}2^3$$
$$= \log_{10}(5 \times 0.25 \times 8)$$
$$= \log_{10}(5 \times 2)$$
$$= \log_{10}10$$
$$= 1$$

2. $\log\frac{11}{5} + \log\frac{14}{3} - \log\frac{22}{15}$

$$= \log\frac{11}{5}\times\frac{14}{3}\times\frac{15}{22}$$
$$= \log 7$$

3. $\log\frac{70}{33} + \log\frac{22}{135} - \log\frac{7}{18}$

$$= \log\frac{70}{33}\times\frac{22}{135}\times\frac{18}{7}$$
$$= \log\frac{10}{3}\times\frac{2}{135}\times 18$$
$$= \log\frac{10}{3}\times\frac{2}{15}\times 2$$

$$= \log\frac{2}{3}\times\frac{2}{3}\times 2$$

$$= \log\frac{2^3}{3^2}$$

$$= 3\log 2 - 2\log 3$$

4. (i) $-\log_5 \log_3 9^{1/10} = -\log_5 \log_3 3^{1/5} = -\log_5(1/5) = 1$

(ii) $\frac{1}{6}\ \log_{\frac{\sqrt{3}}{2}}\left(\frac{64}{27}\right) = \frac{1}{6}\ \log_{\frac{\sqrt{3}}{2}}\left(\frac{\sqrt{3}}{2}\right)^{-6} = -1$

5. $3^{2\log_9 3} = 3^{2\times\frac{1}{2}\log_3 3}$

$$= 3^{\log_3 3}$$

$$= 3^1 = 3$$

6. We have $\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi} = \log_\pi 3 + \log_\pi 4 = \log_\pi 12$

But $\pi^2 < 12 < \pi^3$

$\Rightarrow \quad 2 < \log_\pi 12 < 3$

7. $\log_{1000}\left(x^2\right) = \frac{2}{3}\log_{10} x = \frac{2}{3}y$

8. $\log_{2\sqrt{2}} 32\sqrt[5]{4}$

$$= \log_{(2^{3/2})}(2^5 4^{\frac{1}{5}})$$

$$= \log_{(2^{3/2})}(2^{5+\frac{2}{5}})$$

$$= \frac{2}{3}\frac{27}{5}\log_2 2$$

$$= \frac{18}{5} = 3.6$$

9. $\log_a(ab) = x \Rightarrow \log_a a + \log_a b = x \Rightarrow \log_a b = x - 1$

Now, $\log_b(ab) = \log_b a + \log_b b = \frac{1}{\log_a b} + 1 = \frac{1}{x-1} + 1 = \frac{x}{x-1}$

10. $\frac{1}{y} = \log_{25} 17 = \frac{1}{2}\log_5 17$

and $\quad \frac{1}{x} = \log_5 3 = \frac{1}{2}\log_5 9$

$\therefore \quad \frac{1}{y} > \frac{1}{x} \Rightarrow x > y$

11. $\because y = \frac{1}{2^{\log_x 4}}$

or $\quad \log_2 y = \frac{1}{\log_x 4} \quad (\because x > 0, x \neq 1)$

or $\quad \log_2 y = \log_4 x$

or $\quad \log_2 y = \frac{1}{2}\log_2 x$

or $\quad 2 \log_2 y = \log_2 x$

or $\quad \log_2 y^2 = \log_2 x$

$\therefore \quad x = y^2$

12. $\log_e\left(\frac{a+b}{2}\right) = \frac{1}{2}(\log_e a + \log_e b)$

$\Rightarrow \quad \log_e\left(\frac{a+b}{2}\right) = (\log_e \sqrt{ab})$

$\Rightarrow \quad \frac{a+b}{2} = \sqrt{ab}$

$\Rightarrow \quad a + b - 2\sqrt{ab} = 0$

$\Rightarrow \quad (\sqrt{a} - \sqrt{b})^2 = 0$

$\Rightarrow \quad a = b$

13. Let $a = x - 1, b = x, c = x + 1$

Now, $\log(1 + ac) = \log(1 + (x - 1)(x + 1)) = \log x^2 = 2 \log x = 2 \log b$

$\Rightarrow \quad K = \log b$

14. $\log_k x \cdot \log_5 k = \log_x 5$

$\Rightarrow \quad \frac{\log x}{\log k}\frac{\log k}{\log 5} = \log_x 5$

$\Rightarrow \quad \frac{\log x}{\log 5} = \log_x 5$

$\Rightarrow \quad \log_5 x = \frac{1}{\log_5 x}$

$\Rightarrow \quad (\log_5 x)^2 = 1$

$\Rightarrow \quad \log_5 x = \pm 1$

$\Rightarrow \quad x = 5^{\pm 1}$

$$\Rightarrow \qquad x = \frac{1}{5}, 5$$

15. $3^a = 4 \qquad a = \log_3 4;$

Similarly, $b = \log_4 5$ etc.

Hence $abcdef = \log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8 \cdot \log_8 9 = \log_3 9 = 2$

16. $\dfrac{1+2\log_3 2}{(1+\log_3 2)^2} + \dfrac{(\log_3 2)^2}{(1+\log_3 2)^2} = \dfrac{(1+\log_3 2)^2}{(1+\log_3 2)^2} = 1$

17. $\log_3 5 \cdot \log_{25} 27$

$$= \frac{\log 5}{\log 3}\frac{\log 27}{\log 25} = \frac{\log 5}{\log 3} \times \frac{3}{2}\frac{\log 3}{\log 5} = \frac{3}{2}$$

18. $\log_2 15 \times \log_{1/6} 2 \times \log_3 1/6$

$$= \frac{\log 15}{\log 2} \cdot \frac{\log 2}{\log 1/6} \cdot \frac{\log 1/6}{\log 3}$$

$$= \frac{\log 15}{\log 3} = \frac{\log 3 + \log 5}{\log 3}$$

$$= 1 + \log_3 5 > 1 + 1$$

19. $2^{\log_{(2\sqrt{2})} 15} = 2^{\log_{2^{3/2}} 15} = 2^{\frac{2}{3}\log_2 15} = 2^{\log_2 15^{\frac{2}{3}}} = 15^{\frac{2}{3}}$

20. $4^{5\log_{4\sqrt{2}}(3-\sqrt{6})-6\log_8(\sqrt{3}-\sqrt{2})}$

$$= 4^{2\log_2(\sqrt{3}(\sqrt{3}-\sqrt{2}))-2\log_2(\sqrt{3}-\sqrt{2})}$$

$$= 4^{2\log_2\sqrt{3}}$$

$$= 2^{\log_2(\sqrt{3})^4}$$

$$= 9$$

21. $\left(\dfrac{1}{49}\right)^{1+\log_7 2} + 5^{-\log_{(1/5)}(7)}$

$$= (7^{-2})^{\log_7 14} + 7$$

$$= 7^{\log_7 14^{-2}} + 7$$

$$= 14^{-2} + 7$$

$$= \frac{1}{196} + 7$$

$$= \frac{1373}{196}$$

Exercise 2

1. $$\frac{\log_a x \ \log_b x}{\log_a x + \log_b x}$$

$$= \frac{\dfrac{1}{\log_x a}\,\dfrac{1}{\log_x b}}{\dfrac{1}{\log_x a} + \dfrac{1}{\log_x b}}$$

$$= \frac{\dfrac{1}{\log_x a}\,\dfrac{1}{\log_x b}}{\dfrac{\log_x a + \log_x b}{\log_x a \log_x b}}$$

$$= \frac{1}{\log_x a + \log_x b}$$

$$= \frac{1}{\log_x ab} = \log_{ab} x$$

2. $2 \log (2y - 3x) = \log x + \log y$

$\Rightarrow \quad \log (2y - 3x)^2 = \log xy$

$\Rightarrow \quad 4y^2 + 9x^2 - 12xy = xy$

$\Rightarrow \quad 4y^2 + 9x^2 - 13xy = 0$

$$\Rightarrow \quad 4 + 9\left(\frac{x}{y}\right)^2 - 13\left(\frac{x}{y}\right) = 0$$

$$\Rightarrow \quad 9\left(\frac{x}{y}\right)^2 - 13\left(\frac{x}{y}\right) + 4 = 0$$

$$\Rightarrow \quad \left(9\frac{x}{y} - 4\right)\left(\frac{x}{y} - 1\right) = 0$$

$$\Rightarrow \quad \frac{x}{y} = 1 \text{ or } \frac{4}{9}$$

3. Given $4 \log_a a + 5\log_a b = 0$

$\Rightarrow \quad \log_a b = -4/5$ (1)

Now, $\log_a(a^5 b^4) = 5 + 4 \log_a b = 5 + 4\left(-\frac{4}{5}\right) = 5 - \frac{16}{5} = \frac{9}{5}$

4. $\log_3 c = 3 + \log_3 a$

$\Rightarrow \qquad \log_3 \dfrac{c}{a} = 3$

$\Rightarrow \qquad c = 27a$ (1)

$\log_a b = 2$ and $\log_b c = 2$

$\Rightarrow \qquad \log_a b \cdot \log_b c = 4$

$\Rightarrow \qquad \log_a c = 4$

$\Rightarrow \qquad c = a^4$ (2)

From (1) and (2), we get

$a = 3$ and $c = 81$

From $\log_a b = 2$, we get $b = a^2 = 9$

$\Rightarrow \qquad \dfrac{c}{ab} = 3$

5. $a^x = b,\ b^y = c,\ c^z = a$

$\Rightarrow \qquad x = \log_a b,\ y = \log_b c,\ z = \log_c a$

$\Rightarrow \qquad xyz = (\log_a b)(\log_b c)(\log_c a) = \dfrac{\log b}{\log a}\dfrac{\log c}{\log b}\dfrac{\log a}{\log c} = 1$

6. $\dfrac{1}{1+\log_a bc} + \dfrac{1}{1+\log_b ca} + \dfrac{1}{1+\log_c ab}$

$= \dfrac{\log a}{\log a + \log b + \log c} + \dfrac{\log b}{\log a + \log b + \log c} + \dfrac{\log c}{\log a + \log b + \log c}$

$= 1$

7. $\dfrac{1}{\log_{bc} abc} + \dfrac{1}{\log_{ca} abc} + \dfrac{1}{\log_{ab} abc}$

$\Rightarrow \qquad \log_{abc} bc + \log_{abc} ca + \log_{abc} ab = \log_{abc} a^2b^2c^2 = 2$

8. Let $K = x^{\frac{1}{\log y}+\frac{1}{\log z}} \cdot y^{\frac{1}{\log z}+\frac{1}{\log x}} \cdot z^{\frac{1}{\log x}+\frac{1}{\log y}}$

$\log K = \log x\left[\dfrac{1}{\log y} + \dfrac{1}{\log z}\right] + \log y\left[\dfrac{1}{\log z} + \dfrac{1}{\log x}\right] + \log z\left[\dfrac{1}{\log x} + \dfrac{1}{\log y}\right]$ (1)

using $\log x + \log y + \log z = 0$, we get

$\dfrac{\log x}{\log y} + \dfrac{\log z}{\log y} = -1;\ \dfrac{\log y}{\log x} + \dfrac{\log z}{\log x} = -1$ and $\dfrac{\log x}{\log z} + \dfrac{\log y}{\log z} = -1$

$\therefore \qquad$ RHS of (1) $= -3 \Rightarrow \log_{10} K = -3 \Rightarrow K = 10^{-3}$

9. $N = \dfrac{\log_5 250}{\log_{50} 5} - \dfrac{\log_5 10}{\log_{1250} 5}$

$\Rightarrow \quad N = (3 + \log_5 2)(2 + \log_5 2) (4 + \log_5 2)$

$\Rightarrow \quad N = (\log_5 2)^2 + 5 \log_5 2 + 6 - [(\log_5 2)^2 + 5 \log_5 2 + 4)]$

$\Rightarrow \quad N = 6 - 4 = 2$

10. $\log_{140} 63 = \dfrac{\log_7 7 + 2\log_7 3}{2\log_7 2 + \log_7 7 + \log_7 5}$

$= \dfrac{1+2ac}{2c+1+abc}$

11. $\sqrt[3]{\left(5^{1/\log_7 5} + \dfrac{1}{\sqrt{(-\log_{10} 0.1)}}\right)}$

$= \sqrt[3]{\left(5^{\log_5 7} + \dfrac{1}{\sqrt{(\log_{10} 0.1^{-1})}}\right)}$

$= \sqrt[3]{\left(7 + \dfrac{1}{\sqrt{\log_{10} 10}}\right)} = \sqrt[3]{7+1} = 2$

12. A $\log_{200}5 + B \log_{200}2 = C$

$\therefore \quad$ A log 5 + B log 2 = C log 200 = C $\log(5^2\, 2^3)$

$\therefore \quad$ A log 5 + B log 2 = 2C log 5 + 3 C log 2

$\therefore \quad$ A = 2C and B = 3C

For no common factor greater than 1, C = 1.

$\therefore \quad$ A = 2; B = 3

$\therefore \quad$ A + B + C = 6

13. $\log_3 (\log_2 x) + \log_{1/3} (\log_{1/2} y) = 1$

$\Rightarrow \quad \log_3(\log_2 x) - \log_3(-\log_2 y) = 1$

$\Rightarrow \quad \log_3\left(-\dfrac{\log_2 x}{\log_2 y}\right) = 1$

$\Rightarrow \quad -\dfrac{\log_2 x}{\log_2 y} = 3$

$\Rightarrow \quad xy^3 = 1$

Also, $\quad xy^2 = 9$

$\Rightarrow \quad y = \frac{1}{9}$

$\therefore \quad x = 729$

14. $a > 1, b > 1$

$$2(\log_a c + \log_b c) = 9\log_{ab} c$$

$$\Rightarrow \quad 2\left[\log c\left[\frac{\log b + \log a}{\log a \log b}\right]\right] = 9\frac{\log c}{\log a + \log b}$$

$\Rightarrow \quad 2(\log a + \log b)^2 = 9(\log a)(\log b)$

$\Rightarrow \quad 2(\log a)^2 + 2(\log b)^2 + 4(\log a)(\log b) = 9(\log a)(\log b)$

$\Rightarrow \quad 2\log_b a + 2\log_a b + 4 = 9$

$\Rightarrow \quad 2\log_b a + 2\log_a b = 5$

$\Rightarrow \quad t + \frac{1}{t} = \frac{5}{2}$, where $t = \log_a b$

$\Rightarrow \quad 2t^2 - 5t + 2 = 0$

$\Rightarrow \quad (2t - 1)(t - 2) = 0$

$\Rightarrow \quad t = 1/2, t = 2$

$\Rightarrow \quad \log_a b = 1/2$ or $\log_a b = 2$

Chapter 4

Exercise 1

1. $\log_2(x - 3) = 3$

$\Rightarrow \quad x - 3 = 2^3 = 8$

$\Rightarrow \quad x = 11$

2. $\log_7(x - 2) + \log_7(x + 3) = \log_7 14$

$\Rightarrow \quad \log_7(x - 2)(x + 3) = \log_7 14$

$\Rightarrow \quad x^2 + x - 6 = 14$

$\Rightarrow \quad x^2 + x - 20 = 0$

$\Rightarrow \quad (x - 4)(x + 5) = 0$

$\Rightarrow \quad x = 4, -5$

But for $x = -5$, terms in original equation are not defined.

$\Rightarrow \quad x = 4$ is the only solution.

3. $\log_2(x + 5) - \log_2(2x - 1) = 5$

$$\Rightarrow \quad \log_2 \frac{x+5}{2x-1} = 5$$

$\Rightarrow \quad \dfrac{x+5}{2x-1} = 32$

$\Rightarrow \quad x + 5 = 64x - 32$

$\Rightarrow \quad 63x = 37$

$\Rightarrow \quad x = 37/63$

Also this value of x satisfy terms in original equation.

4. $\log_2 (3x-2) = \log_{1/2} x = \dfrac{\log_2 x}{\log_2 2^{-1}} = \log_2 x^{-1}$

$\Rightarrow \quad 3x - 2 = x^{-1} \Rightarrow 3x^2 - 2x = 1 \Rightarrow x = 1$ or $x = -1/3$.

But $\log_2 (3x-2)$ and $\log_{1/2} x$ are meaningful if $x > 2/3$. Hence $x = 1$.

5. $2\log_{2+\sqrt{3}}\left(\sqrt{x^2+1}+x\right) + \log_{2-\sqrt{3}}\left(\sqrt{x^2+1}-x\right) = 3$

$$\Rightarrow \quad 2\log_{2+\sqrt{3}}\left(\sqrt{x^2+1}+x\right) + \log_{(2+\sqrt{3})^{-1}}\left(\sqrt{x^2+1}-x\right) = 3$$

$$\Rightarrow \quad 2\log_{2+\sqrt{3}}\left(\sqrt{x^2+1}+x\right) - \log_{2+\sqrt{3}}\left(\sqrt{x^2+1}-x\right) = 3$$

$$\Rightarrow \quad \log_{2+\sqrt{3}} \frac{\left(\sqrt{x^2+1}+x\right)^2}{\left(\sqrt{x^2+1}-x\right)} = 3$$

$$\Rightarrow \quad \log_{2+\sqrt{3}}\left(\sqrt{x^2+1}+x\right)^3 = 3$$

$$\Rightarrow \quad \left(\sqrt{x^2+1}+x\right)^3 = (2+\sqrt{3})^3$$

$$\Rightarrow \quad \sqrt{x^2+1}+x = 2+\sqrt{3}$$

$$\Rightarrow \quad x = \sqrt{3}$$

6. The given equality is meaningful if $x - 1 > 0, x - 3 > 0 \Rightarrow x > 3$.

The given equality can be written as

$$\frac{\log(x-1)}{\log 4} = \frac{\log(x-3)}{\log 2}$$

$\Rightarrow \quad \log (x-1) = 2 \log (x-3)$ (log 4 = 2 log 2)

$\Rightarrow \quad (x-1) = (x-3)^2 \Rightarrow x^2 - 7x + 10 = 0$

$\Rightarrow \quad (x-5)(x-2) = 0 \Rightarrow x = 5$ or 2. But $x > 3$ so $x = 5$.

7. $(\log_6 9 + \log_6 4) - \dfrac{\log_3 27}{\log_3 9} = \dfrac{\log_8 x}{2} - \log_8 x$

$$\Rightarrow \quad 2-\frac{3}{2} = -\frac{1}{2}\log_8 x$$

$$\Rightarrow \quad \frac{1}{8} = -\frac{1}{8}\log_8 x$$

$$\Rightarrow \quad x = \frac{1}{8}$$

8. Since the equation can be satisfied only for $x < 0$ hence $\sqrt{x^2} = |x| = -x$,

$$\Rightarrow \quad \sqrt{\log(-x)} = \log(-x) \quad \Rightarrow \quad \log(-x) = [\log(-x)]^2$$

$$\Rightarrow \quad \log(-x)[1-\log(-x)] = 0$$

$$\Rightarrow \quad \text{if } \log(-x) = 0 \Rightarrow -x = 1 \Rightarrow x = -1$$

if $\log_{10}(-x) = 1 \Rightarrow -x = 10 \quad \Rightarrow x = -10$

9. We have

$\log_y x + \log_x y = 2$ (1)

$x^2 + y^2 = 12$ (2)

Let $t = \log_y x$

$(x, y > 0, \text{ and } \neq 1)$

$$t + \frac{1}{t} = 2 \text{ or } (t-1)^2 = 0 \quad \therefore \quad t = \log_y x = 1 \quad \therefore \quad x = y$$

From (2),

$x^2 + x - 12 = 0 \therefore x = -4, 3$

$\therefore \quad x = 3$ only $(-4$ rejected)

$\therefore \quad y = 3$

$\therefore \quad xy = 9$

10. $$\frac{2\log_5(bx+28)}{\log_5(1/5)^2} = -\log_5(12-4x-x^2)$$

$$\Rightarrow \quad bx + 28 = 12 - 4x - x^2$$

$$\Rightarrow \quad x^2 + (b+4)x + 16 = 0$$

Now, $(b+4)^2 = 4(16) \Rightarrow b + 4 = \pm 8$

11. We must have $x - 1 > 0 \Rightarrow x > 1$ (i)

and $5 + 4\log_3(x-1) > 0 \Rightarrow 4\log_3(x-1) > -5 \Rightarrow \log_3(x-1) > -\frac{5}{4}$

$\Rightarrow \quad x - 1 > 3^{-5/4} \Rightarrow x > 1 + 3^{-5/4}$ (ii)

From Eqs. (i) and (ii), $x > 1 + 3^{-5/4}$

$\therefore \quad 5 + 4\log_3(x-1) = 9 \quad \Rightarrow \quad 4\log_3(x-1) = 4$

$\Rightarrow \quad \log_3(x-1) = 1 \quad \Rightarrow \quad x - 1 = 3 \therefore x = 4$

12. $\log_4(3-x) + \log_{0.25}(3+x) = \log_4(1-x) + \log_{0.25}(2x+1)$

$\Rightarrow \quad \log_4(3-x) - \log_4(3+x) = \log_4(1-x) - \log_4(2x+1)$

$\Rightarrow \quad \log_4 (3-x) + \log_4 (2x+1) = \log_4 (1-x) + \log_4 (3+x)$

$\Rightarrow \quad (3-x)(2x+1) = (1-x)(3+x)$

$\Rightarrow \quad 3 + 5x - 2x^2 = 3 - 2x - x^2$

$\Rightarrow \quad x^2 - 7x = 0$

$\Rightarrow \quad x = 0, 7$

Only $x = 0$ is the solution as $x = 7$ is to be rejected.

13. Let $\log_8 x = y$. Then given equation reduces to

$$\frac{1-2y}{y^2} = 3$$

$\Rightarrow \quad 3y^2 + 2y - 1 = 0$

$\Rightarrow \quad 3y^2 + 3y - y - 1 = 0$

$\Rightarrow \quad 3y(y+1) - 1(y+1)$

$\Rightarrow \quad \log_8 x = y = 1/3, -1$

$\Rightarrow \quad x = 2, 1/8$

14. $\log_{16} x = \dfrac{1 \pm \sqrt{1 - 4\log_{16} k}}{2}$. For exactly one solution, $4\ \text{lo}_{16}\, k = 1$

$\therefore \quad k^4 = 16 \therefore k = 2, -2, 2i, -2i.$

15. $\log_3 (5x-2) - 2\log_3 \sqrt{3x+1} = 1 - \log_3 4$

$\Rightarrow \quad \log_3 (5x-2) - \log_3 (3x+1) + \log_3 4 = 1$

$\Rightarrow \quad \log_3\left(\dfrac{(5x-2)(4)}{3x+1}\right) = 1 \quad \Rightarrow \quad \dfrac{(5x-2)(4)}{3x+1} = 3$

$\Rightarrow \quad x = 1$

16. $\log_4 (x+2)^3 + 3 = \log_4 (4-x)^3 + \log_4 (6+x)^3$

$\Rightarrow \quad 3 = \log_4 \left(\dfrac{24 - 2x - x^2}{x+2}\right)^3$

$\Rightarrow \quad \left(\dfrac{24 - 2x - x^2}{x+2}\right)^3 = 4^3$

$\Rightarrow \quad \dfrac{24 - 2x - x^2}{x+2} = 4$

$\Rightarrow \quad x^2 + 6x - 16 = 0$

$\Rightarrow \quad (x-2)(x+8) = 0$

$\Rightarrow \quad x = 2$ (as $x = -8$ does not define original equation)

17. $2\log_{10} a + \log_{10}(a-1) = \log_{10} 2a$

$\therefore \quad a^2(a-1) = 2a$

$\therefore \quad a^2 - a - 2 = 0$

$\Rightarrow \quad (a-2)(a+1) = 0$

$\Rightarrow \quad a = 2, -1$(not possible)

$\therefore \quad a = 2$

18. $\log_5\left(\log_{64}|x| + (25)^x - \frac{1}{2}\right) = 2x$

$\Rightarrow \quad \log_{64}|x| + (25)^x - \frac{1}{2} = (25)^x$

$\Rightarrow \quad \log_{64}|x| = \frac{1}{2}$

$\Rightarrow \quad |x| = 8$

$\Rightarrow \quad x = -8, 8$

19. We have $\log_2(x^2 - x)\log_2\left(\frac{x-1}{x}\right) + (\log_2 x)^2 = 4$

$\Rightarrow \quad [\log_2(x-1) + \log_2 x][\log_2(x-1) - \log_2 x] + (\log_2 x)^2 = 4$

$\Rightarrow \quad [\log_2(x-1)]^2 - (\log_2 x)^2 + (\log_2 x)^2 - 4 = 0$

$\Rightarrow \quad [\log_2(x-1)]^2 = 4$

$\Rightarrow \quad \log_2(x-1) = \pm 2$

$\Rightarrow \quad x - 1 = 4$ or $\frac{1}{4}$

$\Rightarrow \quad x = 5$ or $\frac{5}{4}$

20. $\log_2(25^{x+3} - 1) = 2 + \log_2(5^{x+3} + 1)$

$\Rightarrow \quad \log_2(25^{x+3} - 1) - \log_2(5^{x+3} + 1) = 2$

$\Rightarrow \quad \log_2 \frac{25^{x+3} - 1}{5^{x+3} + 1} = 2$

$\Rightarrow \quad \frac{25^{x+3} - 1}{5^{x+3} + 1} = 2^2$

$\Rightarrow \quad y^2 - 1 = 4y + 4$ (putting $5^{x+3} = y$)

$\Rightarrow \quad y^2 - 4y - 5 = 0$

$\Rightarrow \quad y = -1, 5$

$\Rightarrow \quad 5^{x+3} = 5$

$\Rightarrow \quad x = -2$

21. $\log_4(2 \times 4^{x-2} - 1) + 4 = 2x$

$\Rightarrow \quad \log_4(2 \times 4^{x-2} - 1) = 2x - 4$

$\Rightarrow \quad 2 \times 4^{x-2} - 1 = 4^{2x-4}$

$\Rightarrow \quad 2y - 1 = y^2$ (putting $y = 4^{x-2}$)

$\Rightarrow \quad y^2 - 2y + 1 = 0$

$\Rightarrow \quad y = 1$

$\Rightarrow \quad 4^{x-2} = 1$

$\Rightarrow \quad x = 2$

Exercise 2

1. $\because$ $\log_x 2 \log_{2x} 2 = \log_{4x} 2$

$\therefore$ $\quad x > 0, 2x > 0$ and $4x > 0$ and

$x \neq 1, 2x \neq 1, 4x \neq 1$

$\Rightarrow \quad x > 0$ and $x \neq 1, \frac{1}{2}, \frac{1}{4}$

Then, $\quad \dfrac{1}{\log_2 x} \cdot \dfrac{1}{\log_2 2x} = \dfrac{1}{\log_2 4x}$

$\Rightarrow \quad \log_2 x \,.\, \log_2 2x = \log_2 4x$

$\Rightarrow \quad \log_2 x \,.\, (1 + \log_2 x) = (2 + \log_2 x)$

$\Rightarrow \quad (\log_2 x)^2 = 2$

$\Rightarrow \quad \log_2 x = \pm\sqrt{2}$

$\therefore \quad x = 2^{\pm\sqrt{2}}$

$\therefore \quad x = \{2^{-\sqrt{2}}, 2^{\sqrt{2}}\}$

2. Given equation is

$$\frac{1}{1+\log_2 x} + \frac{\log_2 2x}{2} = -\frac{3}{2}$$

$$\Rightarrow \quad \frac{1}{1+\log_2 x} + \frac{1+\log_2 x}{2} = -\frac{3}{2}$$

Let $1 + \log_2 x = y$

$\Rightarrow \quad \frac{1}{y} + \frac{y}{2} = -\frac{3}{2}$

$\Rightarrow \quad 2 + y^2 + 3y = 0$

$\Rightarrow \quad y = -1$ or -2

$\Rightarrow \quad 1 + \log_2 x = -1$ or -2

$\Rightarrow \quad \log_2 x = -2$ or -3

$\Rightarrow \quad x = 2^{-2}$ or 2^{-3}

3. $\log_{ax} a + \log_x a^2 + \log_{a^2 x} a^3 = 0$

$$\Rightarrow \quad \frac{1}{\log_a ax} + \frac{2}{\log_a x} + \frac{3}{(\log_a a^2 x)} = 0$$

$$\Rightarrow \quad \frac{1}{\log_a a + \log_a x} + \frac{2}{\log_a x} + \frac{3}{(2+\log_a x)} = 0$$

Let $\log_a x = y$. Then equation reduces to

$$\frac{1}{y+1} + \frac{2}{y} + \frac{3}{2+y} = 0$$

$\Rightarrow \quad 6y^2 + 11y + 4 = 0$

$\Rightarrow \quad y = \log_a x = -\dfrac{1}{2}, -\dfrac{4}{3}$

$\Rightarrow \quad x = a^{-4/3}, a^{-1/2}$

4. $\dfrac{10}{3} = 3 + \dfrac{1}{3}$. The given equation is of the form $p + \dfrac{1}{p} = 3 + \dfrac{1}{3} = q + \dfrac{1}{q}$, where p $\neq q$ as $x \neq y$

$\Rightarrow \quad \log_2 x = 3$ and $\log_2 y = 1/3 \Rightarrow x = 2^3$ and $y = 2^{1/3}$ $\therefore$ $x + y = 8 + 2^{1/3}$

5. $\log_3 (\log_2 x) + \log_{1/3} (\log_{1/2} y) = 1$

$\Rightarrow \quad \log_3 (\log_2 x) - \log_3(\log_{1/2} y) = 1$

$\Rightarrow \quad \log_3 (\log_2 (4/y^2)) - \log_3(\log_{1/2} y) = 1$

$\Rightarrow \quad \log_2 (4/y^2) = 3(\log_{1/2} y)$

$\Rightarrow \quad \log_2 (4/y^2) = -3(\log_2 y)$

$\Rightarrow \quad \log_2 (4/y^2) + (\log_2 y^3) = 0 \quad \Rightarrow \quad 4y = 1$

$\Rightarrow \quad y = 1/4 \quad \Rightarrow \quad x = 64$

6. $2^{x+2} . 5^{6-x} = 2^{x^2} . 5^{x^2}$

$\Rightarrow \quad 5^{6-x-x^2} = 2^{x^2-x-2}$

$\Rightarrow \quad (6 - x - x^2) \log_{10} 5 = (x^2 - x - 2) \log_{10} 2$ (base 10)

$\Rightarrow \quad (6 - x - x^2) [1 - \log 2] = (x^2 - x - 2) \log_{10} 2$

$\Rightarrow \quad 6 - x - x^2 = (\log_{10} 2) [(x^2 - x - 2) - x^2 - x + 6]$

$\Rightarrow \quad 6 - x - x^2 = (\log_{10} 2) [4 - 2x]$

$\Rightarrow \quad x^2 + x - 6 = 2 (\log_{10} 2) (x - 2)$

$\Rightarrow \quad (x + 3) (x - 2) = (\log_{10} 4) (x - 2)$

$\Rightarrow \quad$ either $x = 2$ or $x + 3 = \log_{10} 4$

$\Rightarrow \quad x = \log_{10} 4 - 3 = \log_{10}\left(\dfrac{4}{1000}\right) \Rightarrow x = -\log_{10} (250)$

7. $\left(\sqrt{1 + \dfrac{3}{2\log_3 x}}\right)\log_3 x + 1 = 0$

Let $\log_3 x = y$. Then equation reduces to

$$\left(\sqrt{1 + \frac{3}{2y}}\right) y = -1 \Rightarrow \left(1 + \frac{3}{2y}\right) = \frac{1}{y^2} \Rightarrow \frac{2y+3}{2y} = \frac{1}{y^2}$$

$2y^2 + 3y - 2 = 0 \Rightarrow 2y^2 + 4y - y - 2 = 0 \Rightarrow (y + 2) (2y - 1) = 0$

$y = 1/2$ or $y = -2 \Rightarrow x = 3^{1/2}$ (rejected) or $x = 1/9$

8. $(\log_a x^2) \log_a x = (k - 2) \log_a x - k$ (taking log on base a)

Let $\log_a x = t$. Then equation reduce to

$$2t^2 - (k-2)t + k = 0$$

Put D = 0 (for only one solution)

$\Rightarrow \quad (k-2)^2 - 8k = 0$

$\Rightarrow \quad k^2 - 12k + 4 = 0$

$\Rightarrow \quad k = \dfrac{12 \pm \sqrt{128}}{2}$

$\Rightarrow \quad k = 6 \pm 4\sqrt{2}$

9. $7^{\log_7(x^2-4x+5)} = (x-1)$

$\Rightarrow \quad x^2 - 4x + 5 = x - 1$

$\Rightarrow \quad x^2 - 5x + 6 = 0$

$\Rightarrow \quad (x-2)(x-3) = 0 \Rightarrow x = 2$ or $x = 3$

Also we must have $x^2 - 4x + 5 > 0$ and $x - 1 > 0$

$\Rightarrow \quad x > 1$ (as $x^2 - 4x + 5 > 0$ is true for all real numbers)

10. $(\log_{10} x)^2 = \log_{10} 100x$

$(\log_{10}x)^2 = 2 + \log_{10}x$

On putting $\log x = t$, we get

$t^2 = 2 + t$

$\Rightarrow \quad t^2 - t - 2 = 0$

$\Rightarrow \quad t = 2$ or $t = -1$

$\Rightarrow \quad \log_{10}x = 2 \quad$ or $\log_{10}x = -1$

$\Rightarrow \quad x = 100$ or $x = 1/10$

11. $\log_3x \cdot \log_4x \cdot \log_5x = \log_3x \cdot \log_4x + \log_4x \cdot \log_5x + \log_5x \cdot \log_3x$

Let $\log_x3 = p$, $\log_x4 = q$, $\log_x5 = r$

$\Rightarrow \quad \dfrac{1}{pqr} = \dfrac{1}{pq} + \dfrac{1}{qr} + \dfrac{1}{pr}$

$\Rightarrow \quad p + q + r = 1$

$\Rightarrow \quad \log_x3 + \log_x4 + \log_x5 = 1$

$\Rightarrow \quad \log_x60 = 1$

$\Rightarrow \quad x = 60$

12. $\log_{3/4} \log_8 (x^2 + 7) + \log_{1/2} \log_{1/4} (x^2 + 7)^{-1} = -2$

$\Rightarrow \quad \log_{3/4}\left(\dfrac{1}{3}\log_2(x^2+7)\right) - \log_2 \dfrac{\log_2(x^2+7)}{2} = -2$

let $\log_2(x^2 + 7) = t$. Then

$\log_{3/4}\dfrac{t}{3} - \log_2\dfrac{t}{2} + 2 = 0$

$$\Rightarrow \quad \log_{3/4}\frac{t}{3} + 1 - \left(\log_2\frac{t}{2} - 1\right) = 0$$

$$\Rightarrow \quad \log_{3/4}\frac{t}{4} = \log_2\frac{t}{4}$$

$$\Rightarrow \quad \frac{t}{4} = 1$$

$$\Rightarrow \quad t = 4$$

$$\Rightarrow \quad \log_2(x^2 + 7) = 4$$

$$\Rightarrow \quad x^2 + 7 = 16$$

$$\Rightarrow \quad x^2 = 9$$

$$\Rightarrow \quad x = \pm 3$$

13. $5^{\log x} - 3^{\log x - 1} = 3^{\log x + 1} - 5^{\log x - 1}$

$$\Rightarrow \quad 5^{\log x} + 5^{\log x - 1} = 3^{\log x + 1} + 3^{\log x - 1}$$

$$\Rightarrow \quad 5^{\log x} + \frac{5^{\log x}}{5} = 3 \cdot 3^{\log x} + \frac{3^{\log x}}{3}$$

$$\Rightarrow \quad \frac{6 \cdot 5^{\log x}}{5} = \frac{10 \cdot 3^{\log x}}{3}$$

$$\Rightarrow \quad \left(\frac{3}{5}\right)^{\log x} = \left(\frac{3}{5}\right)^2$$

$$\Rightarrow \quad \log_{10} x = 2$$

$$\Rightarrow \quad x = 100$$

14. $\log_{x^2} 16 + \log_{2x} 64 = 3$

$$\Rightarrow \quad \frac{4}{\log_2 x^2} + \frac{6}{\log_2 2x} = 3$$

$$\Rightarrow \quad \frac{2}{\log_2 x} + \frac{6}{1 + \log_2 x} = 3$$

Putting $\log_2 x = t$, we get

$$\frac{2}{t} + \frac{6}{1+t} = 3$$

$$\Rightarrow \quad 2 + 2t + 6t = 3t + 3t^2$$

$$\Rightarrow \quad 3t^2 - 5t - 2 = 0$$

$$\Rightarrow \quad t = -\frac{1}{3}, t = 2$$

$$\Rightarrow \quad \log_2 x = -\frac{1}{3} \text{ or } \log_2 x = 2$$

$$\Rightarrow \quad x = 2^{-1/3} \text{ or } x = 4$$

15. We have $9^{\log_3(\log_e x)} = \log_e x - (\log_e x)^2 + 1$

$\therefore \quad x > 1$

The given equation $2(\log_e x)^2 - (\log_e x) - 1 = 0$

$\therefore \quad \log_e x = \dfrac{-1}{2}, 1$

$\Rightarrow \quad x = e, \dfrac{1}{\sqrt{e}}$ (not possible)

$\Rightarrow \quad x = e$

16. $(3x)^{\log 3} = (4y)^{\log 4}$, $4^{\log x} = 3^{\log y}$

$\Rightarrow \quad (\log 3)(\log 3x) = (\log 4)(\log 4y)$ and $(\log x)(\log 4) = (\log y)(\log 3)$

$\Rightarrow \quad (\log 3)(\log 3 + \log x) = (\log 4)(\log 4 + \log y)$ and

$(\log x)(\log 4) = (\log y)(\log 3)$

$\Rightarrow \quad (\log 3)(\log 3 + p) = (\log 4)(\log 4 + q)$ and

$p(\log 4) = q(\log 3)$ (where $p = \log x$ and $q = \log y$)

$\Rightarrow \quad (\log 3)\left(\log 3 + \dfrac{q \log 3}{\log 4}\right) = \log 4(\log 4 + q)$ (eliminating p)

$\Rightarrow \quad (\log 3)^2 - (\log 4)^2 = \dfrac{(\log 4)^2 - (\log 3)^2}{\log 4} q$

$\Rightarrow \quad q = -\log 4 \Rightarrow \log y = \log 4^{-1} \Rightarrow y = 1/4$

Now $p(\log 4) = q(\log 3)$

$\Rightarrow \quad p(\log 4) = -(\log 4)(\log 3)$

$\Rightarrow \quad p = -\log 3$

$\Rightarrow \quad \log x = \log 3^{-1}$

$\Rightarrow \quad x = 1/3$

17. Let $\log_y x = t \Rightarrow x = y^t$ (1)

Also $x^t = 2$ and $y^{1/t} = 2^4$

$\Rightarrow \quad x = 2^{1/t}$ (2)

and $\quad y = 2^{4t}$ (3)

Putting the values of x and y in (1), we get

$2^{1/t} = 2^{4t^2} \Rightarrow 4t^3 = 1$

$\therefore \quad t = \left(\dfrac{1}{4}\right)^{1/3}$ (4)

using (4) in (2); $x = (2)^{(4)^{1/3}} = 2^{\sqrt[3]{4}}$

using (4) in (3); $y = (2)^{(4)^{2/3}}$

Chapter 5

Exercise 1

1. $1 < \log_2(x-2) \le 2$

$\Rightarrow \quad 2^1 < x-2 \le 2^2$

$\Rightarrow \quad 4 < x \le 6$

2. $\log_2|x-1| < 1$

$\Rightarrow \quad 0 < |x-1| < 2^1$

$\Rightarrow \quad -2 < x-1 < 2$ and $x-1 \ne 0$

$\Rightarrow \quad -1 < x < 3$ and $x \ne 1$

$\Rightarrow \quad x \in (-1, 3) - \{1\}$

3. $\log_3|x| > 2$

$\Rightarrow \quad |x| > 3^2$

$\Rightarrow \quad x < -9$ or $x > 9$

4. $|4-5x| > 2^2 = 4$

$\Rightarrow \quad \left|\frac{5x}{4} - 1\right| > 1$

$\Rightarrow \quad \frac{5x}{4} - 1 > 1$ or $\frac{5x}{4} - 1 < -1$

$\therefore \; x > \frac{8}{5}$ or $x < 0$. So, the solution set $= (-\infty, 0) \cup \left(\frac{8}{5}, \infty\right)$

5. $\log_2 \frac{x-1}{x-2} > 0$

$\Rightarrow \quad \frac{x-1}{x-2} > 2^0$

$\Rightarrow \quad \frac{x-1}{x-2} > 1$

$\Rightarrow \quad \frac{x-1}{x-2} - 1 > 0$

$\Rightarrow \quad \frac{x-1-x+2}{x-2} > 0$

$\Rightarrow \quad \frac{1}{x-2} > 0$

$\Rightarrow \quad x > 2$

6. $\log_2 \frac{x-4}{2x+5} < 1$

$\Rightarrow \quad 0 < \dfrac{x-4}{2x+5} < 2^1$

$\Rightarrow \quad \dfrac{x-4}{2x+5} > 0$ and $\dfrac{x-4}{2x+5} < 2$

$\Rightarrow \quad \dfrac{x-4}{2x+5} > 0$ and $\dfrac{x-4}{2x+5} - 2 < 0$

$\Rightarrow \quad \dfrac{x-4}{2x+5} > 0$ and $\dfrac{-3x-14}{2x+5} < 0$

$\Rightarrow \quad \dfrac{x-4}{2x+5} > 0$ and $\dfrac{3x+14}{2x+5} > 0$

$\Rightarrow \quad x < -5/2$ or $x > 4$ and $x < -14/3$ or $x > -5/2$

$\Rightarrow \quad x \in (-\infty, -14/3) \cup (4, \infty)$

7. $\dfrac{x+2}{x} \geq (0.2)^1 \quad \Rightarrow \dfrac{x+2}{x} \geq \dfrac{1}{5}$.

Multiplying by $5x^2$ (positive),

$5x(x+2) \geq x^2 \Rightarrow 4x^2 + 10x \geq 0 \Rightarrow x \geq 0$ or $x \leq -\dfrac{5}{2}$.

Also $\dfrac{x+2}{x} > 0$

$\Rightarrow \quad x(x+2) > 0 \Rightarrow x < -2$ or $x > 0$

Hence, the solution set is $\left(-\infty, -\dfrac{5}{2}\right] \cup (0, +\infty)$.

8. $\log_3(2x^2 + 6x - 5) > 1$

$\Rightarrow \quad 2x^2 + 6x - 5 > 3^1$

$\Rightarrow \quad 2x^2 + 6x - 8 > 0$

$\Rightarrow \quad x^2 + 3x - 4 > 0$

$\Rightarrow \quad (x-1)(x+4) > 0$

$\Rightarrow \quad x < -4$ or $x > 1$

9. $x^2 - 6x + 12 \leq \left(\dfrac{1}{2}\right)^{-2}$

$\Rightarrow \quad x^2 - 6x + 8 \leq 0$

$\Rightarrow \quad (x-2)(x-4) \leq 0.$

$\Rightarrow \quad x \in [2, 4]$

Also, $x^2 - 6x + 12 > 0$ or $(x-3)^2 + 3 > 0$ which is true for any real x.

Hence $x \in [2, 4]$

10. Let $\log_3 x = y$

$\Rightarrow \quad x = 3^y \quad (1)$

Then given inequality $2\log_3 x - 12\log_x 3 \le 5$ reduces to

$$2y - \frac{12}{y} \le 5$$

$\Rightarrow$ $2y^2 - 5y - 12 \le 0$ (as $x > 1 \Rightarrow y > 0$)

$\Rightarrow$ $(2y+3)(y-4) \le 0$

$\Rightarrow$ $y \in \left[-\frac{3}{2}, 4\right]$

$\Rightarrow$ $-\frac{3}{2} \le \log_3 x \le 4$

$\Rightarrow$ $3^{-3/2} \le x \le 81$

11. $\log_{10} x^2 \ge 0$

$\Rightarrow$ $\log_{10} x^2 \ge \log_{10} 1 \Rightarrow x^2 \ge 1$

$\Rightarrow$ $x \ge 1$ or $x \le -1$.

12. $2^{\log_2 (x-1)} > x + 5$

$\Rightarrow$ $x - 1 > x + 5$

$\Rightarrow$ $-1 > 5$; which is not possible.

13. Taking logarithm with base 5, we have

$x^{\log_5 x} > 5 \Rightarrow (\log_5 x)(\log_5 x) > 1 \Rightarrow (\log_5 x - 1)(\log_5 x + 1) > 0$

$\Rightarrow$ $\log_5 x > 1$ or $\log_5 x < -1 \Rightarrow x > 5$ or $x < 1/5$

Also we must have $x > 0$. Thus $x \in (0, 1/5) \cup (5, \infty)$

14. $(\log_{(0.6)} (0.6)^3) \log_5(5 - 2x) \le 0 \Rightarrow 5 - 2x \le 1 \Rightarrow x \ge 2$ (i)

Also, $5 - 2x > 0$ ...(ii)

From (i) and (ii), we have $x \in [2, 2.5)$

15. $(x+2)(x+4) > 0$ and $x + 2 > 0$

$\Rightarrow$ $x > -2$

Now given inequality can be written as $\log_3 (x+2)(x+4) - \log_3 (x+2) < \log_3 7$

$\Rightarrow$ $\log_3 (x+4) < \log_3 7 \Rightarrow x + 4 < 7$ or $x < 3$

16. $x^2 - 16 \le 4x - 11 \Rightarrow x^2 - 4x - 5 \le 0 \Rightarrow (x-5)(x+1) \le 0 \Rightarrow -1 \le x \le 5$ (i)

Also $x^2 - 16 > 0 \Rightarrow x < -4$ or $x > 4$ (ii)

And $4x - 11 > 0 \Rightarrow x > 11/4$ (iii)

From (i), (ii) and (iii) we have $x \in (4, 5]$

17. $\log_{\sqrt{0.9}} \log_5 (\sqrt{x^2 + 5 + x}) > 0$

$\Rightarrow$ $0 < \log_5 (\sqrt{x^2 + 5 + x}) < 1$

$\Rightarrow$ $1 < (x^2 + 5 + x)^{1/2} < 5$

$\Rightarrow$ $1 < x^2 + 5 + x < 25$

$\Rightarrow \quad x^2 + x - 20 < 0$ (as $x^2 + x + 4 > 0$ for all real x)

$\Rightarrow \quad (x + 5)(x - 4) < 0$

$\Rightarrow \quad x \in (-5, 4)$

18. We have $\log_{1/2}(4 - x) \geq \log_{1/2}2 - \log_{1/2}(x - 1)$

It is defined if $4 - x > 0$ and $x - 1 > 0$ or $1 < x < 4$ (i)

$\Rightarrow \quad \log_{1/2}(4 - x)(x - 1) \geq \log_{1/2}2$

$\Rightarrow \quad (4 - x)(x - 1) \leq 2$

$\Rightarrow \quad x^2 - 5x + 6 \geq 0$

$\Rightarrow \quad (x - 3)(x - 2) \geq 0$

$\Rightarrow \quad x \geq 3$ or $x \leq 2$ (ii)

From (i) and (ii), we get $x \in (1, 2] \cup [3, 4)$

19. $2^{x+2} - 2^{x+3} - 2^{x+4} > 5^{x+1} - 5^{x+2}$

$\Rightarrow \quad 2^x(4 - 8 - 16) > 5^x(5 - 25)$

$\Rightarrow \quad (2/5)^x < 1$

$\Rightarrow \quad x \in (0, \infty)$

20. $\left(\frac{3}{4}\right)^{6x + 10 - x^2} < \frac{27}{64}$

$\Rightarrow \quad \left(\frac{3}{4}\right)^{6x + 10 - x^2} < \left(\frac{3}{4}\right)^3$

$\Rightarrow \quad 6x + 10 - x^2 > 3$

$\therefore \quad x^2 - 6x - 7 < 0$

$\therefore \quad (x + 1)(x - 7) < 0$

Exercise 2

1. The left hand side of the inequality is defined for x's which satisfy the following. $1 - x > 0, x - 2 > 0, 1 - x \neq 1$. Obviously there is no single value for which these inequalities are satisfied. Thus the set of its solutions is empty.

2. $x > 0, \frac{1}{2}\log_2 x - 2\left(\frac{\log_2 x}{2}\right)^2 + 1 > 0$

$\Rightarrow \quad \log_2 x - (\log_2 x)^2 + 2 > 0$

$\Rightarrow \quad (\log_2 x)^2 - \log_2 x - 2 < 0$

Let $\log_2 x = t$

Then $\quad t^2 - t - 2 < 0$

$\Rightarrow \quad (t - 2)(t + 1) < 0$

$\Rightarrow \quad -1 < t < 2$

$\Rightarrow \quad -1 < \log_2 x < 2 \Rightarrow \frac{1}{2} < x < 4$

3. We have

$$2\log_{\frac{1}{4}}(x+5) > \frac{9}{4}\log_{\frac{1}{3\sqrt{3}}}(9) + \log_{\sqrt{x+5}}(2)$$

$$\Rightarrow \quad -\log_2(x+5) > \frac{9}{4}\left(-\frac{4}{3}\right) + \frac{2}{\log_2(x+5)}$$

$$\Rightarrow \quad 3 > \log_2(x+5) + \frac{2}{\log_2(x+5)}$$

Let $\log_2(x+5) = y$

$$\Rightarrow \quad 3 > y + \frac{2}{y}$$

$$\Rightarrow \quad \frac{y^2+2}{y} - 3 < 0$$

$$\Rightarrow \quad \frac{y^2-3y+2}{y} < 0$$

$$\Rightarrow \quad \frac{(y-2)(y-1)}{y} < 0$$

Using sign scheme method, we get

$y \in (-\infty, 0) \cup (1, 2)$

$\therefore \quad \log_2(x+5) \in (-\infty, 0) \cup (1, 2)$

$\therefore \quad (x+5) \in (2^{-\infty}, 2^0) \cup (2^1, 2^2)$

$\therefore \quad (x+5) \in (0, 1) \cup (2, 4)$

$\therefore \quad x \in (-5, -4) \cup (-3, -1)$

4. $\log_3 x - (\log_3 x)^2 \leq \frac{3}{2}\log_{(1/2\sqrt{2})} 4$

$$\Rightarrow \quad \log_3 x - (\log_3 x)^2 \leq -2$$

$$\Rightarrow \quad (\log_3 x)^2 - \log_3 x - 2 \geq 0$$

$$\Rightarrow \quad (\log_3 x - 2)(\log_3 x + 1) \geq 0$$

$\Rightarrow \quad \log_3 x \leq -1$ or $\log_3 x \geq 2$

$\Rightarrow \quad x \leq 1/3$ or $x \geq 9$

5. Given $\log_x(x^2 - 1) \leq 0$

If $x > 1$

$\Rightarrow \quad 0 < x^2 - 1 \leq 1$

$\Rightarrow \quad 1 < x^2 \leq 2$

$\Rightarrow \quad x \in [-\sqrt{2}, -1) \cup (1, \sqrt{2}]$

$\Rightarrow \quad x \in (1, \sqrt{2}]$

If $0 < x < 1 \Rightarrow x^2 - 1 \geq 1 \Rightarrow x^2 \geq 2$

$\Rightarrow \quad x \in (-\infty, -\sqrt{2}\,] \cup [\sqrt{2}, \infty)$

$\Rightarrow \quad x \in \phi$

Thus, $x \in (1, \sqrt{2}\,]$

6. $\log_{0.8}\left(\log_6 \dfrac{x^2+x}{x+4}\right) < 0$

$$\Rightarrow \quad \log_6 \frac{x^2+x}{x+4} > 1$$

$$\Rightarrow \quad \frac{x^2+x}{x+4} > 6$$

$$\Rightarrow \quad \frac{x^2+x}{x+4} - 6 > 0$$

$$\Rightarrow \quad \frac{x^2-5x-24}{(x+4)} > 0$$

$$\Rightarrow \quad \frac{(x-8)(x+3)}{(x+4)} > 0$$

From the sign scheme of $\dfrac{(x-8)(x+3)}{(x+4)}$, $x \in (-4, -3) \cup (8, \infty)$

7. We have, $\log_3 (x^2 - 2) < \log_3 \left(\dfrac{3}{2}|x| - 1\right)$

We must have

$x^2 - 2 > 0, \dfrac{3}{2}|x| - 1 > 0$

and $\quad x^2 - 2 < \dfrac{3}{2}|x| - 1$

$$\Rightarrow \quad x^2 > 2, |x| > \frac{2}{3} \text{ and } 2|x|^2 - 3|x| - 2 < 0$$

$$\Rightarrow \quad |x| > \sqrt{2},\ |x| > \frac{2}{3} \text{ and } (2|x|+1)(|x|-2) < 0$$

$$\Rightarrow \quad |x| > \sqrt{2} \text{ and } |x| < 2$$

$$\Rightarrow \quad \sqrt{2} < |x| < 2$$

$$\Rightarrow \quad x \in (-2, -\sqrt{2}\,) \cup (\sqrt{2}, 2)$$

8. $\log_{(x+1)}(x^2 - 4) > 1$

We must have $x^2 - 4 > 0$, $x + 1 > 0$ and $x + 1 \neq 1$

$\therefore \quad x \in (2, \infty)$ (i)

Case I: $x + 1 > 1 \Rightarrow x > 0$

$\therefore \quad x^2 - 4 > x + 1$

$\therefore \quad x^2 - x - 5 > 0$

$\therefore \quad x \in \left(\frac{1+\sqrt{21}}{2}, \infty\right)$ (ii)

Case II: $(x + 1) \in (0, 1) \Rightarrow x \in (-1, 0)$

$\therefore \quad x^2 - 4 < x + 1$

$\therefore \quad x^2 - x - 5 < 0$

$\therefore \quad x \in \left(\frac{1-\sqrt{21}}{2}, \frac{1+\sqrt{21}}{2}\right)$

Hence, we get no solution from here as $x \in (-1, 0)$.

From (i) and (ii), $x \in \left(\frac{1+\sqrt{21}}{2}, \infty\right)$

9. We have, $f(x) = \log_4 \{\log_5 (\log_3(18x - x^2 - 77))\}$

Since $\log_a x$ is defined for all $x > 0$. Therefore, $f(x)$ is defined if

$\log_5 \{\log_3 (18x - x^2 - 77)\} > 0$ and $18x - x^2 - 77 > 0$

$\Rightarrow \quad \log_3 (18x - x^2 - 77) > 5^0$ and $x^2 - 18x + 77 < 0$

$\Rightarrow \quad \log_3 (18x - x^2 - 77) > 1$ and $(x - 11)(x - 7) < 0$

$\Rightarrow \quad 18x - x^2 - 77 > 3^1$ and $7 < x < 11$

$\Rightarrow \quad 18x - x^2 - 80 > 0$ and $7 < x < 11$

$\Rightarrow \quad x^2 - 18x + 80 < 0$ and $7 < x < 11$

$\Rightarrow \quad (x - 10)(x - 8) < 0$ and $7 < x < 11$

$\Rightarrow \quad 8 < x < 10$ and $7 < x < 11$

$\Rightarrow \quad 8 < x < 10$

$\Rightarrow \quad x \in (8, 10)$.

Hence, the domain of $f(x)$ is $(8, 10)$.

10. We have $\frac{x-1}{\log_3(9-3^x)-3} \le 1$

We must have $9 - 3^x > 0$ or $3^x < 9$ or $x < 2$

We have

$$\frac{(x-1)}{\log_3(9-3^x)-\log_3 27} \le 1$$

$$\Rightarrow \quad \frac{(x-1)}{\log_3\left(\frac{9-3^x}{27}\right)} \le 1$$

$$\Rightarrow \quad (x-1)\cdot \log_{\left(\frac{9-3^x}{27}\right)} 3 \le 1$$

$$\Rightarrow \quad \log_{\left(\frac{9-3^x}{27}\right)} (3^{x-1}) \le 1$$

As $x < 2$, $0 < \dfrac{9-3^x}{27} < 1$

We have $3^{x-1} \ge \dfrac{9-3^x}{27}$

$$\Rightarrow \quad 9\cdot 3^x \ge 9 - 3^x$$

$$\Rightarrow \quad 10\cdot 3^x \ge 9$$

$$\Rightarrow \quad x \ge \log_3 0.9$$

Therefore $x \in [\log_3 0.9, 2)$

11. $2^{\frac{4}{x}-x-3} > 2^0$

$$\Rightarrow \quad \frac{4}{x} - x - 3 > 0$$

$$\Rightarrow \quad \frac{4 - x^2 - 3x}{x} > 0$$

$$\Rightarrow \quad \frac{x^2 + 3x - 4}{x} < 0$$

$$\Rightarrow \quad \frac{(x+4)(x-1)}{x} < 0$$

$$\Rightarrow \quad x \in (-\infty, -4) \cup (0, 1)$$

12. $2(25)^x - 5(10^x) + 2(4^x) \ge 0$

$$\Rightarrow \quad 2(5)^{2x} - 5(5)^x(2^x) + 2(2^{2x}) \ge 0$$

$$\Rightarrow \quad 2\left(\frac{5}{2}\right)^{2x} - 5\left(\frac{5}{2}\right)^x + 2 \ge 0$$

$$\Rightarrow \quad \left[\left(\frac{5}{2}\right)^x - 2\right]\left[2\left(\frac{5}{2}\right)^x - 1\right] \ge 0$$

$$\Rightarrow \quad \left(\frac{5}{2}\right)^x \le \frac{1}{2} \text{ or } \left(\frac{5}{2}\right)^x \ge 2$$

$$\Rightarrow \quad x \le \log_{2.5} 0.5 \text{ or } x \ge \log_{2.5} 2$$

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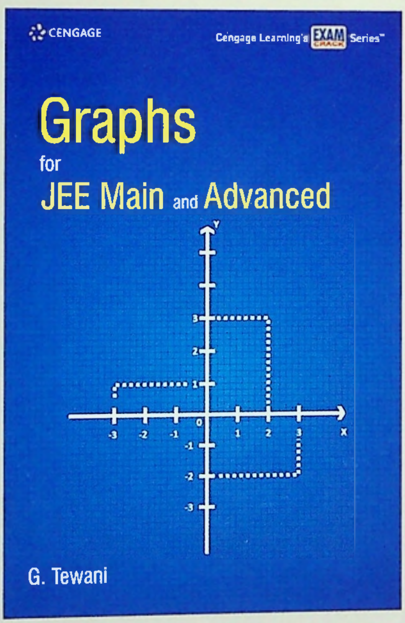

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